# Titles and Abstracts

## OCNMP Conference: Bad Ems, 23–29 June 2024

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# Section I: Regular Talks

1. **Properties and applications of adjoint-symmetries**

   by

   Stephen Anco: *Brock University, Canada*

   **Presenter: Stephen Anco**

   **Abstract:** Symmetries and conservation laws are fundamental coordinate-invariant structures of a given PDE system. They have many important uses, such as exact solutions, conserved norms, invariants, scaling behaviour, detection of linearization and integrability, good discretizations, and so on. Just as there is a natural characteristic form for symmetries, a characteristic form exists for conservation laws by means of multipliers. Specifically, a multiplier is an adjoint-symmetry that satisfies a certain variational condition. The set of infinitesimal symmetries admitted by a given PDE forms a Lie algebra where the bracket is given by the commutator. Similarly, the set of conservation laws admitted by a given PDE has a bracket structure when the PDE is variational (i.e. an Euler-Lagrange equation) or Hamiltonian. Several main questions that will be addressed in this talk are: (1) Can variational structure be detected solely in terms of properties of symmetries and conservation laws? (2) Does a bracket structure for conservation laws exist for non-variational PDEs? (3) What the geometric properties and applications of adjoint-symmetries? Answers come from how symmetries act on adjoint-symmetries. It will be shown that there exist three distinct actions of symmetries on adjoint-symmetries.This leads to dual actions that map symmetries into adjoint-symmetries, from which a natural bracket on adjoint-symmetries can be constructed. One of the dual actions produces multipliers and thereby yields a bracket on multipliers and hence on conservation laws. One other of the dual actions is a Noether operator which reduces to a symplectic operator in the case of variational PDEs. In particular, non-multiplier adjoint-symmetries detect and encode Noether operators and variational structure. This can be used directly to construct a symmetry recursion operator when a PDE is integrable. Finally, adjoint-symmetries have a simple geometrical formulation in terms of invariant one-forms, which is dual to the description of symmetries in terms

of vector fields. These results will be illustrated for physically interesting PDEs, including dissipative systems.

**References:**

[1] S.C. Anco, Symmetry actions and brackets for adjoint-symmetries. I: Main results and applications. Euro. J. Appl. Math (2022), October 19 (on-line), 1–27. *arXiv: 2008.07476*

[2] S.C. Anco, Symmetry actions and brackets for adjoint-symmetries. II: Physical examples. Euro. J. Appl. Math (2022), November 21 (on-line), 1–24. *arXiv: 2208.09994*

[3] S.C. Anco and B. Wang, Geometrical formulation for adjoint-symmetries of partial differential equations. Symmetry 12(9) (2020), 1547 (17 pages). *arXiv: 2009.00779 math-ph*

2. **Orbit method for obtaining rational special function solutions of Painlevé V Hamilton equations**

by
H. Aratyn[1], J.F. Gomes[2], G.V. Lobo[2] and A.H. Zimerman[2]

1: *University of Illinois at Chicago, USA*
2: *Instituto de Fısica Teorica-UNESP, Brazil*

**Presenter: Henrik Aratyn**

**Abstract:** Several extended affine Weyl groups have been identified as Bäcklund symmetry groups of Painlevé equations. A common feature of these symmetry groups is the presence of a commutative group of translations that shift, in a simple manner, the underlying parameters that characterize solutions. We are interested in the role of these abelian subgroups in generating a class of rational solutions of Painlevé equations. Specifically, we will present a construction of special function rational solutions of the Painlevé V Hamilton equations that reside on the orbit of translation operators of a special seed solution of the Painlevé V Hamilton equations. We show how this construction can be straightforwardly extended to include all special function rational solutions of symmetric Painlevé V Hamilton equations beyond those on the orbit and discuss how it leads to degeneracy in the context of the Painlevé V Hamilton formalism.

3. **From Stäckel systems to Painlevé hierarchies**

by
Maciej Błaszak[1], Ziemowit Domański[2] and Krzysztof Marciniak[3]

1: *Adam Mickiewicz University, Poland*
2: *University of Technology, Poznań, Poland*
3: *Linköping University, Sweden*

**Presenter: Maciej Błaszak**

**Abstract:** Among all second order nonlinear integrable ordinary differential equations (ODE's) there are two distinguished classes, playing important roles in modern mathematics and physics. The first class is represented by nonlinear equations of *Stäckel-type*, with an autonomous Hamiltonian representation on a symplectic manifold

$$\frac{d\xi}{dt_r} = X_r(\xi) \equiv \pi dh_r(\xi), \quad r = 1, \ldots, n, \tag{1}$$

which are Frobenius integrable (also known as Liouville integrable in this case)

$$[X_r, X_s] = 0, \quad r, s = 1, \ldots, n.$$

Moreover, the autonomous equations (1) are represented by the isospectral Lax equations

$$\frac{d}{dt_k} L(\lambda; \xi) = [U_k(\lambda; \xi), L(x; \xi)], \quad k = 1, \ldots n. \tag{2}$$

The second class is represented by nonlinear ordinary differential equations of *Painlevé-type*, with a non-autonomous Hamiltonian representation

$$\frac{d\xi}{dt_r} = Y_r(\xi, t) = \pi dH_r(\xi, t), \quad r = 1, \ldots, n, \tag{3}$$

where $t = (t_1, \ldots, t_n)$. The set of $n$ equations (3) constitutes a non-autonomous Painlevé-type system if is Frobenius integrable

$$\frac{\partial X_s}{\partial t_r} - \frac{\partial X_r}{\partial t_s} + \{X_r, X_s\} = 0, \quad r, s = 1, \ldots, n \tag{4}$$

and the system is represented by the isomonodromic Lax representation

$$\frac{d}{dt_k} L(x; \xi, t) = [U_k(x; \xi, t), L(x; \xi, t)] + \frac{\partial U_k(x; \xi, t)}{\partial x}, \quad k = 1, \ldots, n. \tag{5}$$

In this lecture we present a systematic deformation of autonomous Stäckel-type systems to non-autonomous Painlevé-type hierarchies. In particular we construct the infinite hierarchies of Painlevé I ($P_I$), Painlevé II ($P_{II}$), Painlevé III ($P_{III}$) and Painlevé IV ($P_{IV}$).

*I will present results of joint work with* Krzysztof Marciniak *(Linköping, Sweden) and* Ziemowit Domański *(Poznań, Poland)*

**References:**

[1] Błaszak M., Marciniak K., Sergyeyev A., *Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, Rep Math Phys **87** (2021) 249-263

[2] Błaszak M., Marciniak K., Domaǹski Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability*, Stud. Appl. Math. **148** (2022) 1208-1250

[3] Błaszak M., Marciniak K., Domaǹski Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. II. Isomonodromic Lax representation*, Stud. Appl. Math. **149** (2022) 364-415

[4] Błaszak M., *Multi-component Painlevé ODEs and related non-autonomous KdV stationary hierarchies*, Stud. Appl. Math. **151** (2023) 5-34

4. **Affine Weyl groups and non-abelian discrete systems**

by

Irina Bobrova: *Max Planck Institute of the Sciences, Germany*

**Presenter: Irina Bobrova**

**Abstract:** It is well-known that, starting from the Affine Weyl groups (or their extension), one can define a discrete dynamic, by using translation operators (see the paper by M. Noumi and Y. Yamada, CMP, 1998). In fact, a proper extension of a birational representation of the Affine Weyl group acting on a parameter space leads to a discrete system for some dynamical variables. We will introduce a generalization of this construction to a non-abelian case. Moreover, regarding the Painlevé equations, such birational representations naturally arise from the Bäcklund transformations. Since the latter have non-commutative analogs, we will also discuss an application of the Affine Weyl groups for deriving non-abelian versions of the discrete Painlevé equations with additive dynamics. This talk is based on the paper "*Affine Weyl groups and non-abelian discrete systems: an application to d-Painlevé equations*" (*arXiv: 2403.18463*).

5. **Title: Bäcklund transformations in Abelian and non-Abelian nonlinear evolution equations**

by

Sandra Carillo: *Dip. S.B.A.I. Sapienza University of Rome & I.N.F.N. - Sez. Roma1, Gr. IV, Italy*

**Presenter: Sandra Carillo**

**Abstract:** Bäcklund transformations are well known to represent a powerful tool in investigating nonlinear differential equations. In particular, we are concerned about so-called *soliton equations* since they admit *soliton type* solutions. The aim of the present study is twofold since, on one side, we consider the connections which can be established and the induced *structural properties*; on the other side, we consider Bäcklund transformations as a tool to construct solutions, admitted by nonlinear evolution equations. Hence, first of all, we consider the links which can be established among different nonlinear evolution equations via Bäcklund transformations

[2, 3] and references therein. Accordingly, a net of connections among different nonlinear evolution equations is depicted in a *Bäcklund Chart*, as we term such a net of links. The attention is focussed on third order, nonlinear evolution questions in particular, the comparison between the commutative (Abelian) [6, 1] and the noncommutative cases is analyzed. Notably, a richer structure can be observed when the commutativity condition is removed.

Then, solutions, admitted by such equations are presented. Specifically, via Bäcklund transformations, new solutions of matrix modified KdV equation can be constructed. Thus, solutions of matrix mKdV equation [3], obtained on the basis of previous results and in references in [4, 3], are presented. The solutions we obtain can be termed *soliton* solutions since they exhibit the typical behaviour of *solitons*. Indeed, they are shown to represent a generalisation of the corresponding scalar solutions. Finally, some new results [5] as well as some problems, currently under investigation, concerning fifth order nonlinear evolution equations are mentioned.

*Most of the presented results, are part joint research project with Cornelia Schiebold, Sundsvall University, Sweden which involves also, in alphabetical order, M. Lo Schiavo, Rome, E. Porten, Sundsvall, and F. Zullo, Brescia.*

## References

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[2] S. Carillo, M. Lo Schiavo, E. Porten, C. Schiebold, *A novel noncommutative KdV-type equation, its recursion operator, and solitons*, J. Math. Phys., **59**, (4), 14 pp. (2018). doi:10.1063/1.5027481

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[5] S. Carillo, C. Schiebold, F. Zullo *A fifth order nonlinear evolution equation: connection to Caudrey-Dodd-Gibbon-Sawata-Kotera and Möbius induced invariance properties*, in progress, 2024.

[6] B. Fuchssteiner, S. Carillo, *Soliton structure versus singularity analysis: third order completely integrable non linear differential equations in $1+1$-dimensions*, Physica A, **154**, 467–510, (1989). doi:10.1016/0378-4371(89)90260-4

6. **Totally non-negative Pfaffian for solitons in BKP equation**
   by
   Jen-Hsu Chang: *National Defense University, Taiwan*

   **Presenter: Jen-Hsu Chang**

   **Abstract:** The BKP equation is obtained from the reduction of $B$-type in the KP hierarchy under the orthogonal type transformation group for the KP equation. The Schur's $Q$ functions can be used to construct the Tau-function for solitons in BKP equation. Then the totally non-negative Pfaffian can be defined via the Schur's $Q$ functions to obtain non-singular line solitons solution in BKP equation. The N lines solitons interact to form intermediate lines and web-like structures in the near field region. The resonance appearing in this soliton graph in BKP equation can be investigated by the totally non-negative Pfaffian.

   **References:**


[1] Jen-Hsu, Chang, Real Multi-Line Solitons of the BKP Equation, *arXiv:2303.02385*

[2] Xing-Biao Hu and Shi-Hao Li, The partition function of the Bures ensemble as the $T$-function of BKP and DKP hierarchies: continuous and discrete, J. Phys.A: Math. Theor. 50 (2017) 285201 (20pp)

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[6] J.R. Stembridge, Shifted tableaux and projective representations of symmetric groups, Adv. Math. , 74 (1989) pp. 87–134


7. **Common solutions to two Bianchi IX systems**
   by
   Robert Conte: *Centre Borelli, ENS Paris-Saclay, France & The University of Hong Kong, P.R. China*

   **Presenter: Robert Conte**

   **Abstract:** When represented in the synchronous time, the scenario of Belinskii, Khalatnikov and Lifshitz to approach the cosmological singularity is described by a six-dimensional dynamical system. Depending on the choice of the spatial metric tensor, there mainly exist two such systems, whose singlevalued particular solutions are not yet all known, and the goal of this work is to "export" known solutions of one system to unknown cases of the second one. In the first system (mixmaster [1]), all

singlevalued solutions are known in closed form except one four-parameter solution [5, 2]. In the second one (Ryan [6]), only one two-parameter solution is known [4]. We have recently built [3] three more four-parameter singlevalued solutions of limiting cases of the Ryan system, and they happen to match various solutions of the mixmaster, but not the missing one. We conjecture the existence of one four-parameter singlevalued solution of another limiting case of the Ryan system.

**References:**


[1] V.A. Belinskii, E.M. Lifshitz, I.M. Khalatnikov, Oscillatory approach to the singular point in relativistic cosmology, Soviet Physics Uspekhi **13:6** (1971) 745–765.

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[5] A. Latifi, M. Musette and R. Conte, The Bianchi IX (mixmaster) cosmological model is not integrable, Phys. Letters A **194** (1994) 83–92; **197** (1995) 459–460.

[6] Michael P. Ryan, The oscillatory regime near the singularity in Bianchi-type IX universes, Annals of physics **70** (1972) 301–322.


8. **The two-photon coalgebra symmetry and discrete-time systems in quasi-standard form**

Pavel Drozdov: *Università degli Studi di Udine & INFN Sezione di Trieste, Italy*

**Presenter: Pavel Drozdov**

**Abstract:** The coalgebra symmetry method, developed in [2] is one of the most fruitful techniques to produce systematically an N-dimensional Hamiltonian integrable system from 1-dimensional system (see also [1]). This method has been recently adapted for discrete-time systems [4]. In this talk, we give necessary and sufficient conditions for a system of difference equations in the quasi-standard form [3] to admit a coalgebra symmetry with respect to the two-photon Lie algebra. Some integrable examples are discussed within this framework (see also [3]).

**References:**


[1] A Ballesteros, A Blasco, F J Herranz, F Musso, and O Ragnisco. (Super) integrability from coalgebra symmetry: Formalism and applications. J. Phys.: Conf. Ser., 175:012004, June 2009. *doi:10.1088/1742-6596/175/1/012004*.

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9. **Nonlocal invariance and solution iterations for a class of 5th-order integrable evolution equations and their hierarchies**

by
Marianna Euler and Norbert Euler: *International Society of Nonlinear Mathematical Physics, Germany & CIC AC, Mexico*

**Presenter: Marianna Euler**

**Abstract:** We consider the 5th-order Kupershmidt equation and derive a class of related equations that admit nonlocal symmetry invariance. We show how to derive this invariance by using the method of multi-potentialisation. The resulting relations provide nonlocal formulas that act as solution iterations for all the equations in the class, which also applies to the related hierarchies of equations that follow from the equations' recursion operators. Some relevant results are given in [1].

**References:**

[1] Euler M and Euler N, Nonlocal invariance of the multipotentialisations of the Kupershmidt equation and its higher-order hierarchies In: *Nonlinear Systems and Their Remarkable Mathematical Structures*, N Euler (ed), CRC Press, Boca Raton, 317–351, 2018.

10. **Fully-nonlinear evolution equations: the Schwarzian derivative, recursion operators and integrability**

by
Marianna Euler and Norbert Euler: *International Society of Nonlinear Mathematical Physics, Germany & CIC AC, Mexico*

**Presenter: Norbert Euler**

**Abstract:** We discuss the derivation of fully-nonlinear evolution equations of order three. Here one must distinguish between Möbius-invariant equations (that is, equations in one dependent variable that are invariant under the transformation group $SL(2,\mathbb{R})$, and those that are not Möbius-invariant. For the former, the Schwarzian derivative plays an important role for this class of equations, which includes also algebraic nonlinearities [1]. We show how to derive the hierarchy in a simple manner, without calculating the recursion operators for the equations. Using the existence

of recursion operators we furthermore derive a class of third-order fully-nonlinear evolution equations with rational nonlinearities [2] and show how some of the potentialisations of a class of these fully nonlinear third-order equations [3] relate to fully-nonlinear equations of second order with some unexpected observations [4].

**References:**

[1] Euler M and Euler N, On Möbius-invariant and symmetry-integrable evolution equations and the Schwarzian derivative, *Studies in Applied Mathematics*, **143**, 139–156, 2019.

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[3] Euler M and Euler N, Potentialisations of a class of fully-nonlinear symmetry-integrable evolution equations, *Open Communications in Nonlinear Mathematical Physics*, Vol. 4, pp 44–78, 2024 doi:10.46298/ocnmp.13214

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11. **The beauty and power of the complex plane**

by
Athanasios S. Fokas: *University of Cambridge, UK*

**Presenter: Athanasios S. Fokas**

**Abstract:** Two important applications of appropriate deformations to the complex plane will be reviewed: the Unified Transform (also known as the Fokas method), and the development of a new methodology for the asymptotic analysis of the Riemann zeta and related functions.

12. **Symmetry multi-reduction for PDEs with invariant conservation laws**

by
S.C. Anco[1] and M.L. Gandarias[2]

1: *Brock University, Canada*
2: *Cadiz University, Spain*

**Presenter: Maria L. Gandarias**

**Abstract:** A powerful application of symmetries is finding symmetry-invariant solutions of nonlinear differential equations. These solutions satisfy a reduced differential equation in fewer variables given by the invariants of the underlying symmetry.

It is well known that a double reduction of order occurs whenever the starting nonlinear differential equation possesses a conservation law that is invariant with respect to the symmetry.

A broad generalization of the double-reduction method has been derived in a recent work by considering the entire space of invariant conservation laws with respect to a given symmetry. This generalization is able to reduce a nonlinear PDE in 2 variables to an ODE with $m$ first integrals where m is the dimension of the space of invariant conservation laws. Nonlinear PDEs in 3 or more variables can be reduced to an ODE similarly by using an algebra of given symmetries. This algebra does not need to be solvable.

The general method is fully algorithmic. No a priori knowledge of conservation laws of the nonlinear PDE is necessary, and the multi-reduction is carried out in a single step.

In this talk, a summary of the general multi-reduction method will be presented for obtaining invariant solutions. Examples of physically interesting PDEs will be shown.

**References:**

[1] S.C. Anco, Symmetry properties of conservation laws, Int. J. Mod. Phys. B 30(28n29) (2016), 1640004.

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13. **Integrable Hierarchies, Soliton Solutions and Infinite Dimensional Algebras**

by

Y. F. Adans, G. Franca, J. F. Gomes, G. V. Lobo and A. H. Zimerman:
*Instituto de Física Teórica (IFT-Unesp), Brazil*

**Presenter: José Francisco Gomes**

**Abstract:** The construction of Integrable Hierarchies in terms of zero curvature representation provides a systematic construction for a series of integrable non linear evolution equations (flows) which shares a common affine Lie algebraic structure. The integrable hierarchies are then classified in terms of a decomposition of the underlying affine Lie algebra $\hat{G}$ into graded subspaces defined by a grading operator $Q$. In this talk we shall discuss explicitly the simplest case of the affine $sl(2)$ Kac-Moody algebra within the principal gradation given rise to the mkdV hierarchy.

It is known that the positive mKdV sub-hierachy is associated to some positive odd graded abelian subalgebra with elements denoted by $E^{(2n+1)}$. Each of these elements

in turn, defines a time evolution equation according to time $t = t_{2n+1}$. An interesting observation [1] is that for negative grades, the zero curvature representation allows both, even or odd sub-hierarchies. In both cases, the flows are non-local leading to integro-differential equations. It was pointed out in [1] that whilest, the positive and negative odd sub-hierarchies only admit zero vacuum solutions, the negative even admits strictly non-zero vacuum solutions. Soliton solutions can be constructed by gauge transforming the zero curvature from the vacuum into a non trivial configuration (dressing method). In ref [1] the usual vertex operator construction for zero vacuum was extended to include a new parameter describing the constant vacuum configuration. These new soliton solutions were consistently constructed from the deformed vertex operator proposed in [1].

Inspired by the dressing transformation method, we have constructed a Lie algebraic structure to generate systematically Backlund transformation by gauge transformation mapping two soliton configurations [2]. Moreover, gauge transformation was further employed to realise Miura transformation mapping mKdV into KdV flows in [2]. Interesting new results concerns the negative grade sector of the mKdV hierarchy in which a double degeneracy of flows (odd and its consecutive even) of mKdV are mapped into a single odd KdV flow [3].

**References:**


[1] J. F. Gomes, G. Starvaggi França, G. R. de Melo, A. H. Zimerman, "Negative Even Grade mKdV Hierarchy and its Soliton Solutions", J. Phys. A: Math. Theor. 42, (2009),445204, [*arXiv:0906.5579*]

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14. **On a chain of Lie-Poisson coalgebras and their invariants**


by
Giorgio Gubbiotti: *Università degli studi di Milano & INFN sezione di Milano, Italy*


**Presenter: Giorgio Gubbiotti**


**Abstract:** In this talk I will discuss the properties of a chain of Lie algebras, called the $\mathfrak{g}_n$ algebras for $n > 1$, which were recently introduced in [5]. This family of Lie algebras contains known Lie algebras as particular cases: $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{K})$ the two-dimensional special Lie algebra, and $\mathfrak{g}_3 = \mathfrak{h}_6$, the two-photon Lie algebra [7]. In particular, I will present a proof the existence of a generalised Casimir invariant [6] of degree n for every $n > 1$, and show an associated symplectic realisation [4] in N degrees of freedom.

We focus on the proof of the existence of the generalised Casimir. This proof is obtained crafting a representation of the algebra $\bar{\mathfrak{g}}_n = \mathfrak{g}_n/Z(\mathfrak{g}_n)$ in $\mathfrak{sl}_m(\mathbb{K})$, with an appropriate $m = m(n)$, and then proving its invariance under the adjoint action of $\mathrm{SL}_m(\mathbb{K})$. Then, the $N$ degrees of freedom symplectic representation is constructed through the so-called coalgebra symmetry approach [1, 3], see also [2], starting from a one degree of freedom realisation. In particular, we show that the conservation law associated to the nth degree Casimir is non-trivial only for $N \geq n$.

*This is a joint work with Bert van Geemen and Danilo Latini.*

**References:**

[1] A. Ballesteros and O. Ragnisco. "A systematic construction of completely integrable Hamiltonians from coalgebras". In: J. Phys. A: Math. Gen. 31 (1998), pp. 3791–3813.

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[3] Á. Ballestreros, M. Corsetti, and O. Ragnisco. "N-dimensional classical integrable systems from Hopf Algebras". In: Czechoslovak J. Phys. 46 (1996), pp. 1153–1163.

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15. **Solitons in 4d Wess-Zumino-Witten models – Towards unification of integrable systems**

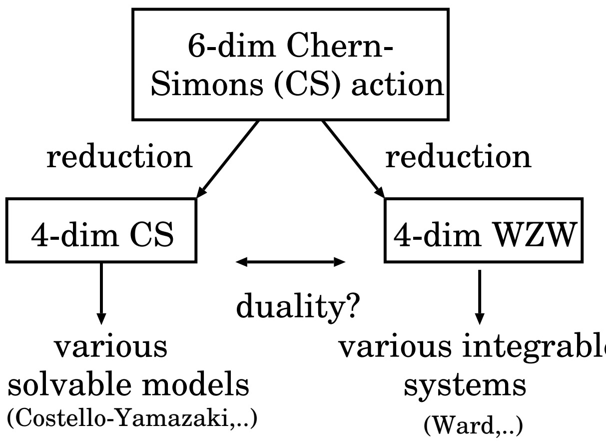
by
Masashi Hamanaka: *Nagoya University, Japan*

**Presenter: Masashi Hamanaka**

**Abstract:** Four-dimensional Wess-Zumino-Witten(4dWZW) models are analogous to the two dimensional WZW models. Equation of motion of the 4dWZW model is the Yang equation which is equivalent to the anti-self-dual Yang-Mills (ASDYM) equation. It is well known as the Ward conjecture that the ASDYM equations can be reduced to many classical integrable systems, such as the KdV eq. and Toda eq. [Ward, Mason-Woodhouse,...]. On the other hand, 4d Chern-Simons (CS) theory has connections to many solvable models such as spin chains and principal chiral models [Costello-Witten-Yamazaki, ...]. Furthermore, these two master equations have been

derived from a 6dCS theory on a twistor space like a double fibration [Costtelo, Bittleston-Skinner]. This suggests a nontrivial duality correspondence between the 4dWZW model and the 4dCS theory.

In this talk, I would like to discuss integrability aspects of the 4dWZW model and construct soliton solutions of it by the Darboux technique [1]. We calculate the action density of the solutions and found that the soliton solutions behaves as the KP-type solitons, that is, the one-soliton solution has localized action (energy) density on a 3d hyperplane in 4-dimensions (soliton wall) [2] and the N-soliton solution describes N intersecting soliton walls with phase shifts [3]. We note that the Ward conjecture holds mostly in the split signature $(+,+,-,-)$ and then the 4dWZW model describes the open N=2 string theory in the four-dimensional space-time. Hence a unified theory of integrable systems can be proposed in this context with the split signature. Our soliton solutions in the 4dWZW models would describe new-type of physical objects (3-brane) in the N=2 string theory [4].

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16. **Formal diagonalisation of integrable systems of evolution equations**
by
Rafael Hernández Heredero[1] and Vladimir V. Sokolov[2]

1: *Universidad Politécnica de Madrid, Spain*
2: *Landau Institute for Theoretical Physics, Russia*

**Presenter: Rafael Hernández Heredero**

**Abstract:** Integrability of multi-component evolution systems

$$u_t^i = \phi^i(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n), \quad i = 1, \ldots, m$$

(where $\mathbf{u} = (u^1, \ldots, u^m)$ and $\mathbf{u}_i = \partial^i \mathbf{u}/\partial x^i$) can be studied using the symmetry approach to integrability. Integrability is usually tested requiring the existence of formal recursion $\mathbf{R}$ and symplectic $\mathbf{S}$ operators of the given system. These objects are matrix pseudo-differential series satisfying the equations

$$\mathbf{R}_t = [\mathbf{\Phi}_*, \mathbf{R}],$$
$$\mathbf{S}_t + \mathbf{S}\,\mathbf{\Phi}_* + \mathbf{\Phi}_*^+\,\mathbf{S} = 0.$$

where $\mathbf{\Phi}_*$ denotes the Fréchet derivative of $\mathbf{\Phi} = (\phi^1, \ldots, \phi^m)$.

Non-degenerate systems, i.e. those whose separant matrix (with entries $\sigma_{ij} = \partial\phi^i/\partial u_n^j$) is invertible and has no multiple eigenvalues at a generic point, are known [1] to be formally diagonalisable in the following sense. Starting from a transformation that diagonalises the separant, a gauge transformation can be found that simultaneously diagonalises $\mathbf{\Phi}_*$ and the recursion and symplectic operators. This splits the relations for $\mathbf{R}$ and $\mathbf{S}$ into a set of $m$ scalar operator equations, yielding a sequence of necessary integrability conditions for the rhs of the system through a procedure well known for scalar equations ($m$=1).

We will explore in this talk the possibility of diagonalising some degenerate systems, introducing the concept of regularly diagonalisable systems. As an illustration, we will deduce explicit integrability conditions and produce a partial classification of integrable systems of the form

$$u_t = v,$$
$$v_t = u_4 + f(u, u_1, u_3, v, v_1).$$

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17. **Solutions to the constant Yang-Baxter equation: additive charge conservation in three dimensions**

by
Jarmo Hietarinta[1], Paul Martin[2] and Eric C. Rowell[3]

1: *University of Turku, Finland*
2: *University of Leeds, UK*
3: *Texas A & M University, USA*

**Presenter: Jarmo Hietarinta**

**Abstract:** The Yang-Baxter equation is difficult to solve even in the constant form $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ and a complete solution is known only for rank two. For further progress it is important to make a meaningful ansatz. Recently Martin and Rowell proposed charge-conservation as an effective constraint (*arXiv:2112.04533*). We explore the results obtained by a slightly different charge-conservation rule.

For more details see *arXiv:2310.03816*.

18. **Generation of Aesthetic Shapes by Integrable Differential Geometry**

by
Kenji Kajiwara: *Kyushu University, Japan*

**Presenter: Kenji Kajiwara**

**Abstract:** In this talk, we consider a class of plane curves called log-aesthetic curves (LAC) and their generalizations which have been developed in industrial design as the curves obtained by extracting the common properties among thousands of curves that car designers regard as aesthetic. We consider these curves in the framework of similarity geometry and characterize them as invariant curves under the integrable deformation of plane curves governed by the Burgers equation. We propose a variational principle for these curves, leading to the stationary Burgers equation as the Euler-Lagrange equation.

We then extend the LAC to space curves by considering the integrable deformation of space curves under similarity geometry. The deformation is governed by the coupled system of the modified KdV equation satisfied by the similarity torsion and a linear equation satisfied by the curvature radius. The curves also allow the deformation governed by the coupled system of the sine-Gordon equation and associated linear equation. The space curves corresponding to the travelling wave solutions of those equations would give generalization of LAC to space curves.

We also consider the surface constructed by the family of curves obtained by the integrable deformation of such curves. A special class of surfaces corresponding to the constant similarity torsion yields quadratic surfaces (ellipsoid, one/two-sheeted hyperboloids and paraboloid) and their deformations, which may be regarded as a generalization of LAC to surface.

We discuss the construction of such curves and surfaces together with their mathematical properties, including integration scheme of the frame by symmetries, and present various examples of curves and surfaces.

*This is a collaborated work with Yoshiki Jikumaru (Toyo University, Japan) and Wolfgang K. Schief (University of New South Wales, Australia).*

19. **The (non)triviality problem for Kontsevich's deformations of Nambu–Poisson brackets**

by
Arthemy V. Kiselev: *University of Groningen, The Netherlands*

**Presenter: Arthemy V. Kiselev**

**Abstract:** Kontsevich's deformations of Poisson structures using directed graphs (1996) were a precursor and by-product of deformation quantisation of the associative product in the algebras of functions on affine finite-dimensional Poisson manifolds (1997). Key to the universal (over manifolds) scheme of infinitesimally deforming the brackets – in such a way that they stay Poisson – is the explicit correspondence between suitable cocycles $\gamma$ in the graph complex (e.g., the tetrahedron $\gamma_3$, pentagon-wheel cocycle $\gamma_5$, their Lie bracket $[\gamma_3, \gamma_5]$, and so on) and Poisson 2-cocycles $Q_\gamma(P)$ for the deformations $d/d\varepsilon(P) = Q_\gamma(P)$. The enigma of Kontsevich's deformations, observed since 1996, is that for all the examples of Poisson brackets $P$ taken so far and for all the graph cocycles $\gamma$ such that the deformations $Q_\gamma(P)$ can be calculated and tested for (non)triviality, the deformations appear to be trivial, $Q_\gamma(P) = [\![P, \vec{X}]\!]$, that is amount to a change of coordinates along integral trajectories of some vector field $\vec{X}$.

We narrow the class of Poisson brackets at hand and refine the calculus of graphs. Namely, to study the mechanism(s) of this 'spontaneous' trivialisation for the class of Nambu-determinant Poisson brackets $P(\varrho, [\boldsymbol{a}])$ with $d-2$ global Casimirs $\boldsymbol{a} = (a_1, \ldots, a_{d-2})$ over dimension $d$, we use a magnifying glass that resolves vertices with $P$ in Kontsevich's graphs to micro-graphs where the Casimirs are distinct as vertices from the Levi–Civita symbols of Jacobian determinants in $P(\varrho, [\boldsymbol{a}])$. We thus detect the (uniqueness of) trivialisation of Kontsevich's tetrahedral flow $Q_d^{\gamma_3}$ over $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ by 1-vectors $X_d^{\gamma_3}$ modulo Hamiltonian vector fields. Using the micro-graph calculus, we explore (obstructions to) the dimensional step $d \mapsto d+1$. We conclude that the trivialisation, $Q_d^\gamma = [\![P, X_d^\gamma]\!]$, for Nambu–Poisson brackets and graph cocycles $\gamma_3$, $\gamma_5$, $\gamma_7$, etc., is due to mechanism(s) which are different from the combinatorial topology that worked in deformation quantisation.

*(Joint work in progress with F. M. Schipper, M. S. Jagoe Brown, and R. Buring.)*

20. **Lagrangian formalism and Noether-type theorems for second-order delay ordinary differential equations**

by
Roman Kozlov: *Norwegian School of Economics, Norway*

**Presenter: Roman Kozlov**

**Abstract:** A Lagrangian formalism for variational second-order delay ordinary differential equations (DODEs) is developed. The Noether operator identity for a DODE is established, which relates the invariance of a Lagrangian function with the appropriate variational equations and the conserved quantities. The identity is used to formulate Noether-type theorems that give the first integrals for DODE with symmetries. Relations between the invariance of the variational second-order DODEs and the invariance of the Lagrangian functions are also analyzed. Several examples illustrate the theoretical results.

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21. **Isoclinism and relative central extensions of a pair of Lie superalgebras**

by
Tofan Kumar Khuntia: *National Institute of Technology Rourkela, India*

**Presenter: Tofan Kumar Khuntia**

**Abstract:** In the study of Lie algebras and their representation theory, an extension $e$ of a Lie algebra $\mathfrak{g}$ is an enlargement of $\mathfrak{g}$ by another Lie algebra $\mathfrak{h}$ . In particular, central extensions of Lie algebras and Lie superalgebras have been studied extensively by many authors in this field. Recently the concept of relative central extension has been defined for a pair of Lie algebras and Lie superalgebras. In this work, we have shown the relation among the relative central extensions in an isoclinism family of a particular relative central extension. Here, at first we have defined the notion of isoclinism on the relative central extensions of a pair of Lie superalgebras. Then in particular, we have figured out the concept of isomorphism in the equivalence class of isoclinisms on the relative central extensions of a pair of Lie superalgebras.

22. **Wind-wave interaction in finite depth: linear and nonlinear approaches, blow-up and soliton breaking in finite time, integrability perspectives**

by
A. Latifi, M. A. Manna, R. A. Kraenkel: *Qom University of Technology, Iran*

**Presenter: Anouchah Latifi**

**Abstract:** This work is an analytical investigation of the evolution of surface water solitary waves in Miles and Jeffreys' theories of wind wave interaction in water of finite depth. The analytic approach is essential for further numerical investigations due to the scale of energy dissipation near coasts. Indeed, the scale of energy dissipation is of the order of a micrometre, which requires $10^{25}$ mesh nodes to produce

correct predictions on scales of 100 km. Hence, no pure numerical modelling of this problem without recourse to theoretical developments has a chance of succeeding. Although many works have been conducted based on Miles and Jeffreys' approach, only a few studies (mainly by M.A. Manna, R.A. Kraenkel, A. Latifi and collaborators [1-4]) are carried out on finite depth. This study is divided into two major sections. The first corresponds to the surface water waves in a linear regime. In this section, Miles' theory of wave amplification by wind is extended to the case of finite depth. The dispersion relation provides a wave growth rate depending on depth. A dimensionless water depth parameter, depending on the depth and a characteristic wind speed, induces a family of curves representing the wave growth as a function of the wave phase velocity and the wind speed. Our theoretical results are in good agreement with the data from the Australian Shallow Water Experiment and the data from the Lake George experiment.

In the second part of this study, Jeffreys' theory of wave amplification by wind is extended to the case of finite depth, where the fully nonlinear anti-dissipative Serre-Green-Naghdi (SGN) equation is derived. The anti-dissipation occurs due to the continuous transfer of wind energy to water surface waves. We find the solitary wave solution of the system, with an increasing amplitude under the action of the wind. This continuous increase in amplitude leads to the soliton breaking and blow-up of the surface wave in finite time for infinitely large asymptotic space. This dispersive, anti-dissipative and fully nonlinear phenomenon is equivalent to the linear instability at infinite time. The theoretical blow-up time is calculated based on actual experimental data.

By applying an appropriate perturbation method, the SGN equation yields a Korteweg–de Vries–Burger-type equation (KdV-B), combining weak nonlinearity, dispersion, and anti-dissipation. We show that the continuous transfer of energy from wind to water results in the growth of the KdV-B soliton's amplitude, velocity, acceleration, and energy over time while its effective wavelength decreases. This phenomenon differs from the classical results of Jeffreys' approach due to finite depth. Again, blow-up and breaking occur in finite time. These times are calculated and expressed for soliton- and wind-appropriate parameters and values. These values are measurable in usual experimental facilities. A detailed analysis of the breaking time is conducted regarding various criteria. The validity of these criteria is examined by comparing these times to the experimental results.

In the end, in the context of wind-forced waves in finite depth, the nonlinear Schrödinger equation is derived, and for weak wind inputs, the Akhmediev, Peregrine and Kuznetsov-Ma breather solutions are obtained.

The extension of this work aims to find an integrable system of coupled wind-wave equations in finite depth by including the air dynamics in the problem and using the formalism recently developed by A.S. Fokas and A. Latifi [5-8] after a multiscale analysis.

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## 23. On dynamics of solitary waves solutions for the generalized Camassa-Holm equation

by
Xiao-Chuan Liu: *Xi'an Jiaotong University, P.R. China*

**Presenter: Xiao-Chuan Liu**

**Abstract:** The following generalized Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 2\kappa u_x + \frac{(p+1)(p+2)}{2} u^p u_x = \frac{p(p-1)}{2} u^{p-2} u_x^3 + 2pu^{p-1} u_x u_{xx} + u^p u_{xxx}$$

with $p \in \mathbb{Z}^+$ appears originally in a paper by Hakkaev and Kirchev (Comm. PDE, 2005) as a special case of the family of CH-type equations:

$$u_t - u_{xxt} + a(u)_x = \big(\frac{1}{2} b'(u) u_x^2 + b(u) u_{xx}\big)_x$$

for $a(u) = 2\kappa u + \frac{p+2}{2} u^{p+1}$ and $b(u) = u^p$. Later, Anco *et al* in (DCDEA, 2015) derived it in terms of one Hamiltonian structure of the classical CH equation. This higher-order equation admits smooth solitary wave solutions when $\kappa \neq 0$ and peaked solitary wave solutions $c^{1/p} e^{-|x-ct|}$ $(c > 0)$ if $\kappa = 0$.

In this talk, I will report our recent results on the dynamical behavior of these solitary wave solutions.

24. **Algebraic curves as a source of separable multi-Hamiltonian systems**

by
Krzysztof Marciniak: *Linköping University (Campus Norrköping), Sweden*
*This talk is an effect of a joint work with prof. Maciej Błaszak, Poznań, Poland.*

**Presenter: Krzysztof Marciniak**

**Abstract:** In [8] Sklyanin noted that any Liouville integrable system (that is a set of $n$ Hamiltonians $h_i$ in involution on a $2n$-dimensional manifold $M$) separates in a given canonical coordinate system $(\lambda, \mu) \equiv (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ if and only if there exists $n$ separation relations of the form

$$\varphi_i(\lambda_i, \mu_i, h_1, ..., h_n) = 0, \qquad i = 1, \ldots, n \tag{1}$$

Alternatively, one can treat the relations (1) as an algebraic definition of $n$ commuting, by construction, Hamiltonians $h_i$ on $M$. The canonical variables $(\lambda, \mu)$ are then by construction separation variables for all $h_i$. This shift of view yields a powerful way of generating separable Hamiltonian systems.

In this talk I focus on the important subclass of separations relations (1) where all $\varphi_i$ are the same, $\varphi_i = \varphi$. In such a case the relations (1) can be interpreted as $n$ copies of the algebraic curve on the $\lambda$-$\mu$ plane

$$\varphi(\lambda, \mu, h_1, ..., h_n) = 0. \tag{2}$$

In this talk I will develop the idea of constructing various types of finite-dimensional integrable and separable Hamiltonian systems from *parameter-dependent* planar algebraic curves. I will focus on two particular constructions. First, I will consider separable systems generated by algebraic curves depending on a set of $n + n$ rather than $n$ parameters. Each such curve leads then to two distinct integrable Hamiltonian systems. I will demonstrate that these systems are related by a Stäckel transform [4, 5, 7, 3] and also how solutions of these two systems are related by reciprocal (multi-time) transformations. I also specify these results to the case of Stäckel systems. Then, I will consider algebraic curves (2) depending on $n + N$, $N > 0$, parameters and having a certain block-type structure. These curves leads to families of integrable and separable Hamiltonian systems that can be related with each other by a finite-dimensional analogue of Miura maps, which yields in turn their multi-Hamiltonian formulation. These results generalize the particular results obtained earlier in [1] and in [6].

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25. **First integrals and $\mathcal{C}^\infty$-structures for ordinary differential equations**

by
C. Muriel, A. J. Pan-Collantes, A. Ruiz: *University of Cádiz, Spain*

**Presenter: Maria Concepcion Muriel**

**Abstract:** The notion of $\mathcal{C}^\infty$-structure for involutive distributions of vector fields has been recently introduced [6] as an extension of the concept of solvable structure [1, 2, 9]. This generalization is inspired by the extension of Lie point symmetries to $\mathcal{C}^\infty$-symmetries for ODEs [3].

In this talk we focus on the applications of $\mathcal{C}^\infty$-structures to find exact solutions for ordinary differential equations (ODEs) [7, 8]. The knowledge of a $\mathcal{C}^\infty$-structure for a given nth-order ODE allows us to construct a sequence of n completely integrable Pfaffian equations, defined on spaces of decreasing dimensions. In the special case when the $\mathcal{C}^\infty$-structure is a solvable structure, the Pfaffian equations can be locally integrated by quadrature.

We study the relationships between $\mathcal{C}^\infty$-structures, first integrals, and integrating factors, extending some of the results obtained by using $\mathcal{C}^\infty$-symmetries for second-order ODEs [4]. The role of Jacobi last multipliers [5] in the integrability by quadrature of the Pfaffian equations associated to a $\mathcal{C}^\infty$-structure is also investigated.

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26. **Lagrangian multiform structure of discrete, semi-discrete and 'generating' PDE type KP systems**

by
Frank W Nijhoff: *University of Leeds, UK & Shanghai University, P.R. China*

**Presenter: Frank W Nijhoff**

**Abstract:** A brief review of KP type systems will be presented, which comprises all of its manifestations: fully discrete (as octahedral lattice equation), semi-discrete (as differential-difference equations with either two discrete and one continuous or two continuous and one discrete variable) and fully continuous KP type systems in terms of Miwa variables). In fact, the 'generating PDE' of the KP hierarchy constitutes a new result. Furthermore, based on these structures, the Lagrangian 3-form structure for these systems will be discussed, providing their variational description. Time allowing, the reduction to the 2-form structure of the Gel'fand- Dikii hierarchy will be discussed and the corresponding 'double zero' structure.

27. **A review of two and three-dimensional superintegrable systems that are linearized through hidden symmetries**

by
Maria Clara Nucci: *University of Bologna, Italy*

**Presenter: Maria Clara Nucci**

**Abstract:** A major drawback of Lie's method for ordinary differential equations is that it is useless when applied to systems of $n$ first-order equations, e.g. Hamiltonian equations, because they admit an infinite number of Lie symmetries, and there is no systematic way to find even one-dimensional Lie symmetry algebra, apart from trivial groups like translations in time admitted by autonomous systems. However, in [15] it was remarked that any system of $n$ first-order equations could be transformed into an equivalent system where at least one of the equations is of second order. Then, the admitted Lie symmetry algebra is no longer infinite dimensional, and

hidden symmetries of the original system could be retrieved. Consequently, in [15] hidden symmetries of the Kepler problem were determined by this method. Also, in [18] the well-known linearization of the Kepler problem, as well as the linearity of generalizations of the Kepler problem with and without drag were determined by means of hidden symmetries.

Such hidden symmetries are neither generalized symmetries (see e.g.[21]) nor those considered in [6], which are just *symmetries of the Hamiltonian, in the sense that they are canonical transformations where both positions and momenta change, and that leave the Hamiltonian function unchanged*.

In [19], it was shown that a two-dimensional superintegrable system [20], such that the corresponding Hamilton-Jacobi equation does not admit the separation of variables in any coordinates, can be transformed into a linear third-order equation by means of hidden symmetries.

In [10], several examples of classical superintegrable systems in two-dimensional Euclidean space [9, 22] were shown to possess hidden symmetries leading to their linearization, and it was conjectured that all classical superintegrable systems in two-dimensional spaces have hidden symmetries that make them linearizable.

In [11], nineteen classical superintegrable systems in two-dimensional non-Euclidean spaces [12, 1, 2] were shown to possess hidden symmetries leading to linearity.

In [16], maximally superintegrable Hamiltonian systems in three-dimensional Euclidean space [7, 8] were also linearized by means of their hidden symmetries, and it was conjectured that three-dimensional minimally superintegrable systems may be similarly linearizable.

In [17], minimally superintegrable Hamiltonian systems in three-dimensional Euclidean space [7] were shown to possess hidden symmetries leading to their linearization.

In [5] fifteen three-dimensional classical minimally superintegrable systems in a static electromagnetic field [13, 14, 3, 4] are shown to possess hidden symmetries leading to their linearization, and consequently the corresponding subsets of maximally superintegrable subcases are also linearizable. These results are strengthening the conjecture that all three-dimensional minimally superintegrable systems are linearizable by means of hidden symmetries, even in the presence of a magnetic field.

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28. **On Korteweg-de Vries and reciprocal moving boundary problems**

by
Colin Rogers: *The University of New South Wales, Australia*

**Presenter: Colin Rogers**

**Abstract:** Classes of moving boundary problems of Stefan-type for the Korteweg-de Vries equation are shown to be exactly solvable in terms of classical Airy functions. The procedure involves application of the Miura transformation to a class of exact Airy-type similarily solutions derived via a Painlevé II reduction of the mKdV equation. Reciprocal transformations are then applied, to obtain, in turn, Airy-type solution of moving boundary problems both for a nonlinear evolution equation of magma theory and a novel reciprocal Korteweg-de Vries equation which incorporates a source term.

29. **On the calculation of exact solutions to a family of general Liénard equations**

by
A. Ruiz and C. Muriel: *University of Cádiz, Spain*

**Presenter: Adrian Ruiz**

**Abstract:** In this talk the general Liénard second-order ordinary differential equation is considered. Such a equation is a particular case of the Levinson-Smith equation and constitutes a generalization of the Rayleigh-type equation, the classical Liénard equation and the quadratic Liénard equation.

By using the techniques based on $\lambda$-symmetries, the general exact solution for a family of general Liénard equations is obtained. Remarkably, the obtained family includes equations that only admits $\partial_t$ as Lie point symmetry and for which the classical Lie symmetry method fails in the obtainment of the corresponding exact general solution.

30. **Linear control system on Lie supergroup and its controllability**

by
Aroonima Sahoo: *National Institute of Technology Rourkela, India*

**Presenter: Aroonima Sahoo**

**Abstract:** In this work, we deal with the linear control system $\Sigma$ having the state-space as Lie supergroup $\mathcal{G}$. The dynamic of the system consists of a drift vector field and control vectors. The drift vector field lies in the normalizer of Lie superalgebra $\mathfrak{g}$ corresponding to Lie supergroup $\mathcal{G}$ whereas the control vectors are the left-invariant vector fields of $\mathcal{G}$. Many literature studies are available on the study of control system on Lie group but here we are trying to generalize these notions into Lie supergroup which is a supermanifold having group structure. Here, we establish the notions of Controllability in case of Lie supergroup and using the tools of supergeometry we develope the rank condition analogous to Lie algebra rank condition to study the Controllability of such dynamical systems.

31. **Long-lived vortices in a sheared flow of the intermediate geostrophic regime**

by
A. Nakamula[1], K. Obuse[2], Nobuyuki Sawado[3], K.Shimasaki[3], Y.Suzuk[3] and K. Toda[4]

1: *Kitasato University, Japan*
2: *Okayama University, Japan*
3: *Tokyo University of Science, Japan*
4: *Toyama Prefectural University, Japan*

**Presenter: Nobuyuki Sawado**

**Abstract:** The term 'long-life' or 'long-lived' has gained much attention in various aspects of physics. Solitons and solitary waves play major roles in (abnormal) long-lived objects. In the theory of the integrable system, stability is guaranteed by the existence of conserved quantities, and if there are of infinite number, the lives become enormous. In the field theoretical context, topological solitons, i.e., skyrmions, gauged vortices, or monopoles, are stable objects of collective motions whose stability is guaranteed in terms of their topological nature [1]. On the other hand, oscillons [2] or an ensemble of Q-balls [3] are notable examples of long-life without topological origin. In the planetary atmosphere, there are a large number of phenomena concerning vortices of nonlinear dynamics. Jupiter's Red Spot is considered a solitary wave because of its extraordinarily long-life. For a deeper understanding, several dynamical regimes have been proposed so far [4], and the zonal flows were regarded as a kind of guide rail to inhibit the transport of eddies across the flows. However, we have found the zonal flows play a major role in the long-life, since they are a source for keeping the conserved quantities such as the energy, or the enstrophy.

We use a simple model on the beta-plane in the intermediate geostrophic regime proposed by Williams, Yamagata [5], and Flierl [6]. This W-Y-F equation has several advantages:

(i) The model is based on dynamics of the Korteweg-de Vries (K-dV) or related known integrable models and then, it enables incorporates many fruits from the studies of the integrable systems.

(ii) The simple, 2-dimensional computation brings us superior mathematical intuition for the system.

(iii) The model integrates several independent effects concerning the longevity and the confluence of vortices; the synergy has not been examined yet.

W-Y-F equation has connections to a 2+1-dimensional nonlinear wave equation by Zakharov-Kuznetsov (Z-K) [7], which may be seen as a model for the cyclonic shear in the uniform background flow. There is one parameter family of the circular symmetric solutions. In terms of the similarity of the equations, these solutions of the Z-K equation are good initial guesses for the anti-cyclonic solutions. By imposing the background shear flow, we can get stable solitary waves of anti-cyclonic solutions. For several strengths and the direction of flows, we examine the life of the vortices. Also, the W-Y-F equation possesses an advection term that realizes the merger of the vortices; it is a quite distinct feature from the standard integrable systems. This effect certainly contributes stability of the vortices. We perform the numerical analysis, especially the synergy of these effects, and explain the origin of the longevity.

For the numerical analysis, we adopted an explicit method with 4th order finite difference stencil for the spatial derivatives and evolved in time with the classical Runge-Kutta 4th order scheme. In the presentation, we also discuss why the zonal flows can support the existence of those extraordinal long-lived vortices and how they interact with each other. A detail of our analysis is presented in [8].

*This work is supported by JSPS KAKENHI Grant Number JP20K03278, and also JP23K02794.*

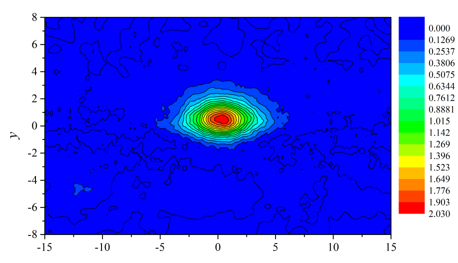


**Figure 1.** The solution of Red Spot in the W-Y-F equation.

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32. **The differential geometry of the multi-dimensionally consistent TED equation**

by
Wolfgang K Schief: *University of New South Wales, Australia*

**Presenter: Wolfgang K Schief**

**Abstract:** The notion of multi-dimensional consistency has proven to be central in both the algebraic and geometric theory of discrete integrable systems. In this talk, we present a natural differential-geometric interpretation of a highly symmetric 4+4-dimensional dispersionless integrable differential equation. The multi-dimensional consistency of this (TED) equation arises naturally in the geometric context and, in fact, guarantees the consistency of the geometric construction leading to the underlying system of compatible TED equations. Emphasis is put on algebraic aspects so that those who are less familiar with the necessary geometric concepts may still be able to follow the discussion.

33. **Non-autonomous deformations of integrable soliton hierarchies**

by
Błażej Szablikowski: *Adam Mickiewicz University, Poland*

I will present results of joint work with *Maciej Błaszak* (Poznań, Poland) and *Krzysztof Marciniak* (Linköping, Sweden).

**Presenter: Błażej Szablikowski**

**Abstract:** Research in the area of integrable non-autonomous hierarchies of soliton type is quite rare compared to research on integrable autonomous hierarchies. In our work we want to partially fill this gap. Similar research program in the case of finite-dimensional integrable systems, followed by series of papers, was initiated in [1].

Starting with an autonomous integrable hierarchy of commuting systems

$$u_{t_n} = K_n(u, u_x, u_{xx}, \dots) \qquad n \in \mathbb{N}, \qquad [K_i, K_j] = 0, \tag{1}$$

we are interested in deforming it to the time-dependent hierarchy

$$u_{t_n} = \mathbb{K}_n(t_1, \dots, t_n; u, u_x, \dots) \qquad n \in \mathbb{N}, \tag{2}$$

where each vector field $\mathbb{K}_n$ explicitly depends on its own time $t_n$ and other times, $t_1, \ldots, t_{n-1}$, associated with lower order members of the hierarchy. To preserve integrability, existence of common multi-time solutions, and commutativity ($u_{t_i,t_j} = u_{t_j,t_i}$), we need to require Frobenius conditions, which in the case of the hierarchy (2) reduce to the system in the triangular form

$$\frac{\partial \mathbb{K}_j}{\partial t_i} + [\mathbb{K}_i, \mathbb{K}_j] = 0 \qquad i < j, \tag{3}$$

as $\frac{\partial \mathbb{K}_i}{\partial t_j} = 0$ for $i < j$.

We attack the problem of constructing the non-autonomous deformations (2) by reformulating the system of Frobenius conditions (3) as an initial-value problem on an arbitrary Lie algebra and by providing formal solution to this problem. However, applying this solution to the trivial Lie algebra given by (1) leads to the trivial uninteresting result. To overcome this obstacle, we extend this trivial algebra over additional master symmetries $\sigma_i$ (idea proposed in [2]) and thus we work with the so-called hereditary Lie algebra

$$[K_n, K_m] = 0, \qquad [\sigma_n, K_m] = (\alpha m + \rho - 1)K_{n+m}, \qquad [\sigma_n, \sigma_m] = \alpha(m - n)\sigma_{n+m}.$$

As result, we provide, in this setting, nontrivial and (general) solutions for the constructing of the integrability preserving non-autonomous deformations (2) of the integrable autonomous hierarchies (1). We illustrate our theory with examples of well-known hierarchies, starting with the Korteweg-de Vries hierarchy. Additionally, we present construction of (isomondromic) Lax representations for derived non-autonomous hierarchies (2).

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34. **Traveling wave solutions for a static version of the Zakharov-Kuznetsov equation via the Painlevé analysis**


by
A. Nakamula[1], K. Obuse[2] N. Sawado[3] K. Shimasaki[3] Y. Suzuki[3] and Kouichi Toda[4,5]

1: *Kitasato University, Japan*
2: *Okayama University, Japan*
3: *Tokyo University of Science, Japan*
4: *Toyama Prefectural University, Japan*
5: *Keio University, Japan*

**Presenter: Kouichi Toda**

**Abstract:** The Williams-Yamagata-Flierl (WYF) equation is one of the best candidates for the great red spot (GRS) on the surface of Jupiter. The WYF equation is a simple model on the $\beta$-plane in the intermediate geostrophic regime proposed by Williams and Yamagata [1], and Flierl [2]. The WYF equation has several advantages: (i) The model is based on dynamics of the Korteweg-de Vries (KdV) or related known integrable models, and then, it incorporates many fruits from the studies of the integrable systems. (ii) The simple, 2-dimensional computation brings us superior mathematical intuition for the system. iii) The model integrates several independent effects concerning the longevity and the confluence of vortices; the synergy has yet to be examined.

The WYF equation has also connections to a $(1+2)$-dimensional nonlinear wave equation by Zakharov-Kuznetsov (ZK) [3], which may be seen as a model for the cyclonic shear in the uniform background flow. Unfortunately, the WYF equation possesses no conserved quantities [4], thus we must conclude that it is entirely distinct from any known integrable system. Nonetheless, for the longevity of the GRS, an integrable nature in the underlying nonlinear dynamics is of the dominant role. The ZK equation can be regarded as the mimics of the WYF equation and has stable vortex solutions. Therefore, we think it is worthwhile to research the integrable property to comprehend the physics of the GRS. The ZK equation originally was the plasma model with a uniform magnetic field and has been studied as a two-dimensional extension of the well-known KdV equation. The equation for $\phi = \phi(t, x, y)$

$$\frac{\partial \phi}{\partial t} + 2\,\phi\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\nabla^2 \phi\right) = 0 \tag{1}$$

possesses meta-stable solitary vortices, where $\nabla^2 := \partial_x^2 + \partial_y^2$. The single soliton is stable in a dynamical sense. However, especially in the scattering of the two solitons with dissimilar heights, the taller soliton becomes much taller. At the same time, the shorter one turns out to be much shorter and radiates ripples. The ZK equation possesses the solutions propagating in a specific direction with uniform speeds. Here we set the direction in the positive $x$ orientation with the velocity $c$. And then, solutions to Eq.(1) keeping circular symmetry satisfy the equation for $\Phi_c = \Phi_c(r)$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\left(\frac{\mathrm{d}\Phi_c}{\mathrm{d}r}\right)\right\} = c\,\Phi_c - \Phi_c^2\,, \tag{2}$$

where $r := \sqrt{(x - c\,t)^2 + y^2}$, with the boundary condition $\Phi_c \to 0$ as $r \to \infty$ [5].

In this talk, we will derive traveling solutions for Eq.(2) by the Painlevé analysis [6] and introduce the relationship between the traveling wave and the longevity of the GRS.

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## 35. Singularities and growth of higher order discrete equations

by
Ralph Willox: *The University of Tokyo, Japan*

**Presenter: Ralph Willox**

**Abstract:** For birational maps on $\mathbf{C}^2$ a full classification in terms of their singularities and possible degree growths is known both in the autonomous case [1] as well as in the non-autonomous case [2]. For higher dimensional maps however, besides a set of conjectures due to Silverman [3] which effectively restrict the degree growth of integrable maps to polynomials of degree $N(=$ the dimension of the space), there is not much known about the interplay between the possible degree growths of integrable birational maps and their singularity structures.

In this talk I will explain on several examples how integrable maps on $\mathbf{C}^N$ $(N > 2)$ with degree growth that is faster than quadratic can be obtained through coupling with (lower order) linearizable maps and that, generically, such maps possess unconfined singularities which is something that is known to be impossible in the $\mathbf{C}^2$ case.

This talk is based on a forthcoming paper in collaboration with Takafumi Mase, Alfred Ramani and Basil Grammaticos [4].

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36. **Exact Solitary Wave Solutions for a Coupled gKdV-NLS System**

by
Stephen C. Anco[1], James Hornick[2], Sicheng Zhao[3] and Thomas Wolf[1]

1: *Brock University, Canada*
2: *McMaster University, Canada*
3: *Queen's University, Canada*

**Presenter: Thomas Wolf**

**Abstract:** We study a coupled gKdV-NLS system $u_t + \alpha u^p u_x + \beta u_{xxx} = \gamma(|\psi|^2)_x$, $i\psi_t + \kappa\psi_{xx} = \sigma u\psi$ with nonlinearity power $p > 0$, which has been introduced in the literature to model energy transport in an anharmonic crystal material [1,2]. There is a strong interest in obtaining exact solutions describing frequency-modulated solitary waves $u = U(x-ct)$, $\psi = e^{i\omega t}\Psi(x-ct)$, with wave-speed $c$, and modulation frequency $\omega$. Some solutions have been found for $p = 1$ (KdV) in [1], while for $p = 2$ (mKdV), no exact solutions were found [2]. Nothing has been done for $p \geq 3$.

We derived exact solutions for $p = 1, 2, 3, 4$, starting from the travelling wave ODE-system satisfied by $U$ and $\Psi$. The method is new: (i) obtain first integrals by use of multi-reduction symmetry theory [3]; (ii) apply a hodograph transformation which leads to a triangular system; (iii) introduce an ansatz for polynomial solutions of the base ODE; (iv) characterize conditions under which solutions yield solitary waves; (v) solve an algebraic system for the unknown coefficients under those conditions.

The resulting solitary waves exhibit a wide range of features: bright/dark peaks; single/multi-peaked; zero/non-zero backgrounds.

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37. **Stability of KdV $n$-solitons**

by
Derchyi Wu: *Academia Sinica, Taiwan*

**Presenter: Derchyi Wu**

**Abstract:** Using the inverse scattering theory, we prove an orbital stability theorem of KdV $n$-solitons with explicit phase shifts in the soliton region with cones around the $x$-axis and lines determined by bound states of the KdV $n$-solitons removed.

38. **Supersymmerties of a new supersymmetric dispersionless system and the classifications**

by
Ruoxia Yao[1] and Senyue Lou[2]

1: *Shaanxi Normal University, P.R. China*
2: *Ningbo University, P.R. China*

**Presenter: Ruoxia Yao**

**Abstract:** Infinitely many higher order super symmetries with arbitrary functions of a new integrable supersymmetric dispersionless system are constructed via symmetry analysis and the bosonization method along with dimensional analysis method and the operation of covariant derivative operator under the scheme of fermion field. The higher order symmetries are not commutable and the commutators yield new higher order symmetries. Importantly, different types of super-symmetries are obtained respectively in the cases of same ranks (or called dimension) of the quentities referred to the monomials in the general symmetry forms, and then we give the classifications of them.

39. **Stability of doubly periodic patterns by using Floquet analysis**

by
H. M. Yin[1], J. H. Li[2], Z. Zheng[3], K. S. Chiang[4], and K. W. Chow[1]

1: *University of Hong Kong, Hong Kong*
2: *Nanjing University of Information Science & Technology, P.R. China*
3: *City University of Hong Kong, Hong Kong*

**Presenter: H. M. Yin**

**Abstract:** In this talk, we explore doubly periodic solutions expressible in terms of the Jacobi elliptic functions for the nonlinear Schrödinger equation. These solutions are of particular interest due to their resemblance to the doubly periodic patterns observed in experiments in fluid mechanics and optics. The stabilities of these doubly periodic wave profiles in the focusing regime are studied computationally by using two different approaches. Firstly, we analyze the stability using Floquet theory, examining the eigenvalues of the monodromy matrix. Instability is indicated when the modulus of these eigenvalues exceeds unity. This is verified by numerical simulations with input patterns of different periods. Initial patterns associated with larger eigenvalues will disintegrate faster due to stronger instability. Secondly, the formation of these doubly periodic patterns from a tranquil background is scrutinized. Doubly periodic profiles are generated by perturbing a continuous wave with a single Fourier

mode, with or without the additional presence of random noise. The predicted instability of these doubly periodic profiles aligns excellently with that derived from Floquet analysis.

40. **Elliptic solitons and related problems**
by
Da-jun Zhang: *Shanghai University, P.R. China*

**Presenter: Da-jun Zhang**

**Abstract:** Apart from the classical solitons that are composed by usual exponential type plane wave factors, there exist "elliptic solitons" which are composed by the Lamé-type plane wave factors and expressed using Weierstrass functions. Recently, we found vertex operators to generate tau functions for such type of solitons. We also established an elliptic scheme of direct linearization approach. In this talk, I will briefly review recent progress on elliptic solitons and also will introduce some related problems.

## Section II: Public Talks

1. **Geophysical flows and waves**
by
Adrian Constantin: *University of Vienna, Austria*

**Presenter: Adrian Constantin**

**Abstract:** Using a few examples, it is explained how the interaction between measurements, mathematics and computer simulations enables insight into climate-relevant natural phenomena.

2. **The two 'big bangs' of our mental evolution**
by
Athanasios S. Fokas: *University of Cambridge, UK*

**Presenter: Athanasios S. Fokas**

**Abstract:** Visual perception will be used to show that every conscious experience is preceded by an unconscious process. In particular, in visual perception, which is achieved via the deconstruction of a given percept followed by its reconstruction, about a third of second after an unconscious reconstruction, the unconscious informs consciousness of the given percept. At this moment, the first 'big bang' takes place: awareness. Many of our evolutionary predecessors possess consciousness. So why

do we differ from them qualitatively? Many scholars have highlighted language as the key difference between us and other creatures possessing consciousness. In my opinion, this is not entirely correct. Instead, I propose that we possess a predisposition to construct real versions of our mental images and their unconscious forms, or to assign to them specific symbols. I label the emerging constructions or symbols, metarepresentations. This is the second 'big bang' of or mental evolution, which in addition to language, includes the metarepresentations of mathematics, computers, technology, and arts. A painting of Kandinsky will be used to illustrate the meta-representation of arts.

## Section III: On Open Access Publishing

1. **Episciences overlay journals and their place in the open access landscape**
by
Raphaël Tournoy: *Center for Direct Scientific Communication, France*

**Presenter: Raphaël Tournoy**

**Abstract:** In the traditional academic publishing landscape, journals generally charge readers for access to their content. The cost of access is borne either by library and institutional subscriptions, or by the readers themselves. Over the years, this model has allowed commercial publishers to increase their subscription costs and profits, while being neither efficient nor fair to the academic community.

Another model has emerged, Gold Open Access, which allows free access to content for readers on condition that authors agree to pay an Article Processing Charge (APC). While this model makes it possible to achieve open access to publications, it is not more equitable and even less viable in the long term for authors, libraries and funding bodies.

However, another publication model has emerged in the academic community, Diamond Open Access. This publication model allows researchers to publish free of charge while giving readers free access to the content.

Episciences is a platform for publishing diamond open access scientific journals. Created in 2013 and initiated by a mathematician, it is open to all countries and languages, with no access or publication fees. Episciences publishes overlay journals and, to do so, relies on existing open science infrastructures. The content of publications is hosted in open repositories (such as arXiv, HAL, Zenodo, bioRxiv, medRxiv). Episciences and its overlay journals offer a cost-effective publication system for the profit of researchers and open science.

Led by academics, the platform is supported by French academic funds. Based on preprint servers, the overlay model and a dedicated editorial and support team, Episciences increases the transparency of the editorial workflow and enables researchers to regain control of their publication resources and workflows.