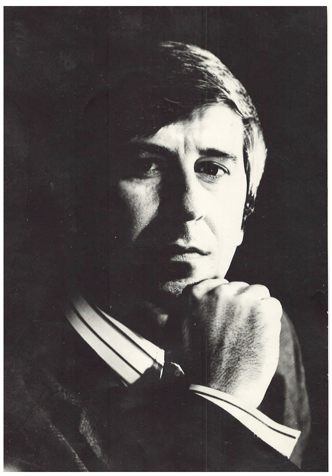
Book of Abstracts

The 2nd ISNMP Conference: Bad Ems, 28 June–4 July 2026

Edited by Norbert Euler

This conference is dedicated to Wilhelm Fushchych and Jarmo Hietarinta, for their seminal contributions in Nonlinear Mathematical Physics: in memory of Wilhelm on his 90th anniversary, and in celebration of Jarmo on his 80th birthday.



Wilhelm I. Fushchych

(18.12.1936 – 07.04.1997)

Jarmo Hietarinta

(born 08.07.1946)

This conference is organised by the *International Society of Nonlinear Mathematical Physics* (ISNMP): a non-profit association and learned society registered and located in Bad Ems, Germany.

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Contact Email Address: Dr.Norbert.Euler@gmail.com

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Below we list the Titles and Abstracts of the Talks in three Session, namely the *Regular Session*, the *Open Problems Session* and the *Student Session*.

[*Click on a name in the Index below to skip directly to that specific abstract*]:

Regular Session Index [Click on a name for the Title & Abstract]

Open Problems Session Index [Click on a name for the Title & Abstract]

Student Session Index [Click on a name for the Title & Abstract]

Regular Session

1. **Title:** To be announced
   by
   Stephen Anco: Brock University, Canada

   Abstract: To be announced.

2. **Volterra lattices and solutions of symmetric Painlevé IV equations**
   by
   H. Aratyn: University of Illinois at Chicago, USA

   Collaborators: Y. F. Adans, J. F. Gomes, G. V. Lobo

   Abstract: We present an explicit construction of rational solutions to the symmetric Painlevé IV equations based on one-to-one identification of the orbits of square-roots of the translation operators of the $A_2^{(1)}$ affine Weyl algebra with the Volterra lattices with special boundary conditions.

3. **On a non-commutative q-PVI system**
by
Irina Bobrova: Technische Universität Berlin, Berlin, Germany

Abstract: We consider a non-commutative analogue of the celebrated q-PVI system introduced in the paper by Jimbo and Sakai. This system later appeared in Sakai's classification of discrete Painlevé equations, where surface theory was applied. Our non-commutative analogue is derived from a birational representation of the extended affine Weyl group of type $D_5^{(1)}$, which we postulated. To obtain this birational representation, we extend Sakai's surface theory and apply it to the non-commutative q-PVI system. In this talk, we discuss a non-commutative generalisation of Sakai's surface theory and its application to the non-commutative q-PVI system.

4. **Bäcklund transformations and non-linear soliton equations: some recent results**
by
Sandra Carillo: Dip. S.B.A.I. Sapienza University of Rome & I.N.F.N., Italy

Collaborators: Cornelia Schiebold and Federico Zullo

Abstract: An excursus on some recent results concerning applications of Bäcklund transformations in the study of soliton questions is provided. In particular, joint results with Cornelia Schiebold, and Federico Zullo, are considered. The focus is on the admitted recursion operator as well as on the properties preserved under Bäcklund transformations. Then, the focus is on the Hamiltonian and bi-Hamiltonian structure admitted by equations which are linked via Bäcklund transformations. Third order and fifth order soliton equations and the Bäcklund transformations which link them are considered. The results can be extended to the corresponding hierarchies. Notably, also in the non-commutative case, crucial properties are preserved via Bäcklund transformations. Results and new perspectives both in the commutative as well as in the non-commutative one are presented.

5. **Lagrangian Bonnet surfaces in $\mathbb{CP}^2$**
by
Robert Conte: LRC MESO, Centre Borelli, France & The University of Hong Kong

Collaborator: Ma Hui

Abstract: Bonnet surfaces [2, 4] are six-parameter surfaces in $\mathbb{R}^3(c)$ whose metric is a particular Chazy C_{VI} function [3], an algebraic transform [1] of the sixth Painlevé function. For Lagrangian surfaces in the two-dimensional complex space forms $\widetilde{\mathbb{M}}^2(4\kappa)$ [5], we define the Bonnet problem and we solve it by the method of

Bonnet. The resulting Lagrangian Bonnet surfaces exist for any sectional curvature κ and their metric is also a particular Chazy C_{VI}.

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6. **A Mathematica package for the symmetry approach to integrability**

by

Rafael Delgado López: Universidad Politécnica de Madrid, Spain

Collaborator: Rafael Hernández Heredero

Abstract: We are interested on the symmetry approach to integrability. In the case of multi-component evolution systems, $u^i_t = \phi^i(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ $(i = 1, \ldots, m$, $\mathbf{u} = (u^1, \ldots, u^m)$, $\mathbf{u}_i = \partial^i \mathbf{u}/\partial x^i)$, integrability under the symmetry approach means the existence of formal recursion $\mathbf{R}$ and symplectic operators $\mathbf{S}$ such that

$$\mathbf{R}_t = [\mathbf{\Phi}_*, \mathbf{R}]\,, \quad \mathbf{S}_t + \mathbf{S}\mathbf{\Phi}_* + \mathbf{\Phi}_*^+\mathbf{S} = 0\,, \quad \mathbf{\Phi} = (\phi^1, \ldots, \phi^m)\,.$$

$\mathbf{\Phi}_*$ is the Fréchet derivative of $\mathbf{\Phi}$, a matrix of differential operators

$$\mathbf{\Phi}_*[i,j] = \sum_{l=0}^{n} \frac{\partial \phi^i}{\partial u^j_l} D^l\,, \qquad D = \frac{d}{dx}\,.$$

However, formal recursion $\mathbf{R}$ and symplectic $\mathbf{S}$ operators are matrix pseudo-differential series,

$$\mathbf{R} = \sum_{l=-r}^{\infty} \mathbf{R}_l D^{-l}\,, \qquad \mathbf{S} = \sum_{l=-s}^{\infty} \mathbf{S}_l D^{-l}\,.$$

We are implementing the algebra of differential and pseudo-differential operators on a Mathematica package on the jet space, so that integrability conditions can be recovered beyond the limits of manual computation. The goal is also not restricting the functions to polynomials but allowing for general functions.

The package is coded as a set of substitution rules acting on passive Mathematica operators (**pd** for partial derivative, **td** for total derivative),... Some of these rules can be applied via instructions like **applyTD** or **applyTD3**. We have a semi-automatic workflow, so that we can leave most of the computation to automatic algorithms. And at the same time complex manual solutions can be added when needed.

References

[1] Mikhailov A V, Shabat A B and Sokolov V V, Symmetry Approach to Classification of Integrable Equations, *What is integrability?* Ed. V.E. Zakharov, Springer Series in Nonlinear Dynamics, Springer-Verlag, 115–184, 1991.

7. A vector Hirota bilinear framework for integrable systems

by

Laurent Delisle: LyRIDS, ECE Engineering School, OMNES Education, Paris, France

Abstract: We investigate the integrable structure and soliton dynamics of a coupled modified Korteweg–de Vries (cmKdV) system with a real symmetric coupling matrix. We introduce a vector reformulation of Hirota's bilinear formalism in which both the bilinear equations and their solutions are expressed directly at the vector level, rather than through a component-wise construction. This formulation preserves the intrinsic structure of the coupled system and provides a compact framework for multi-component nonlinear wave dynamics. Within this approach, we construct explicit one-, two-, and three-soliton solutions in closed vector form and recover the three-soliton condition directly at the vector level, confirming consistency with integrability. The method enables a unified treatment of focusing, defocusing, and mixed-sign regimes. In particular, for indefinite coupling, it reveals the existence of nontrivial vector ground states, leading to soliton solutions on non-zero backgrounds. These results highlight the structural advantages of the vector bilinear approach and open perspectives for the study of more general nonlinear excitations in multi-component integrable systems.

8. Discrete Painlevé Equations with Constraints: Their Geometry and Symmetry

by

Anton Dzhamay: Beijing Institute of Mathematical Sciences and Applications, PR China

Collaborators: Y. Shi, A. Stokes and R. Willox

Abstract: Discrete Painlevé equations are discrete dynamical systems on families of Sakai surfaces that admit actions of extended affine Weyl groups. The Painlevé dynamics is generated by the actions of translations or quasi-translations. The Sakai classification of discrete Painlevé equations is complete on the level of surfaces, as well as on the level of the symmetry groups that generate the dynamics in the generic case. However, there are non-generic cases, i.e. discrete Painlevé equations with constraints, in which the true symmetry group is a proper subgroup of the generic one for its type. We consider a class of examples where constraints correspond to, algebraically, setwise stabilizers of some subset of simple roots, and geometrically, to existence of configurations of nodal curves. Such stabilizers can be described explicitly in terms of generators and relations following the techniques developed by B. Brink and R. Howlett. In this way we can obtain discrete Painlevé equations with non-simply laced affine Weyl symmetry groups, as we illustrate by considering some examples on the $\left(A_0^{(1)}\right)^{**}$ Sakai surface family with the $W\left(E_8^{(1)}\right)$ generic symmetry group.

9. **On fully-nonlinear and quasilinear 5th-order symmetry-integrable evolution equations invariant under the Möbius transformation**
by
Marianna Euler: International Society of Nonlinear Mathematical Physics, Germany & CIC AC, Mexico

Collaborator: Norbert Euler

Abstract: We identify nonlinear evolution equations of order five that are both symmetry-integrable and invariant under the Möbius (or projective) transformation. Those 5th-order evolution equations are of the form

$$u_t = u_x \Phi(S, S_x, S_{xx}),$$

where S is the Schwarzian derivative, which is the 3rd-order invariant of the projective transformation. We show that there exist only three fully-nonlinear equations of this type that are symmetry-integrable. The quasilinear 5th-order case is also discussed in some detail.

10. **On Sequences of fully-nonlinear and quasilinear evolution equations**
by
Norbert Euler: International Society of Nonlinear Mathematical Physics, Germany & CIC AC, Mexico

Collaborator: Marianna Euler

Abstract: We identify some sequences of evolution equations based on their Lie-Bäcklund symmetry and Lie point symmetry properties. In particular we discuss

the odd-order fully-nonlinear sequence of evolution equations

$$u_t = \left(u_{(2k+3)x}\right)^{-\frac{k+1}{k+2}} \qquad k = 0, 1, 2, 3, \ldots$$

which was proposed in our paper [1]. Furthermore we introduce some new sequences of quasilinear evolution equations and discuss several open problems related to such type of sequences.

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11. **The hidden symmetries of Yang-Mills theory in $(3+1)$-dimensions [1]**
by

Luiz Agostinho Ferreira: Instituto de Física de São Carlos; IFSC/USP; Universidade de São Paulo, São Carlos-SP, Brazil

Collaborator: H. Malavazzi

Abstract: We show that classical, non-supersymmetric, Yang-Mills theories coupled to spin-1/2 and spin-0 matter fields, in $(3+1)$-dimensional Minkowski space-time, possess an exact integrability structure with an infinite number of conserved charges in involution. Such an integrability lives on the space of non-abelian electric and magnetic charges, and is based on flat connections in generalized loop spaces, presenting an R-matrix, and Sklyanin relation. We present two novel symmetries of Yang-Mills theories. The first one are global transformations generated by the infinity of conserved charges under the Poisson brackets. The gauge and matter fields, as well as Wilson lines and fluxes have interesting transformation laws under such a global symmetry. The second one corresponds to symmetries of the integral Yang-Mills equations, which lead to the conserved charges. They generate an infinite dimensional group, where the elements are holonomies of connections on the loop space. The conserved charges are gauge invariant, and so in the case of QCD they are color singlets, and perhaps are not confined. Therefore, the hadrons may carry such charges. Our results open up the way for the construction of non-perturbative methods for Yang-Mills theories.

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12. **A dissipative Westervelt's equation: symmetry analysis and hidden variational structure**

by
M.L. Gandarias: Cadiz University, Spain

Collaborators: S.C. Anco, A.P. Márquez, T.M. Garrido

Abstract: Propagation of sound waves in a compressible medium has several important applications where nonlinear and dissipative effects are relevant. Examples are parametric arrays in water and in air, under water imaging, musical acoustics of brass instruments, sonochemistry, quality control and characterization of materials, and bio-medical devices. Especially significant is ultra-sound imaging in human tissue. A simple mathematical 1D model is given by a dissipative version of Westervelt's equation describing the pressure fluctuation. Symmetries and conservation laws are intrinsic, fundamental aspects of wave equations. Their existence is not precluded by dissipative and nonlinear effects. The present work is devoted to illustrating some of these developments for the dissipative Westervelt equation:

- Lie point symmetries of the dissipative Westervelt equation,
- conservation laws of the dissipative Westervelt equation,
- construction of the potential system,
- potential Lie point symmetries,
- potential conservation laws,
- variational structure,
- a recursion operator,
- higher–order symmetries and higher-order conservation laws.

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13. **Darboux Transformations, Quasi-determinants and the Sasa-Satsuma equation**
by
Claire Gilson: School of Mathematics and Statistics, University of Glasgow, UK

Abstract: The Sasa-Satsuma equation is a 3rd order integrable evolution equation exhibiting soliton solutions. In this talk I will examine some other solutions including rational solutions and solutions in terms of quasi determinants aiming at a non-commutative version. The work builds on that of Nimmo and Yilmaz and that of Bandelow and Akhmediev amoungst others.

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14. **New solitonic solutions for higher grading integrable hierarchies**
by
Jose Francisco Gomes: IFT-Unesp, Brazil

Abstract: A generalized framework to accommodate higher graded integrable hierarchies is proposed, primarily extending the conventional algebraic formalism. This approach utilizes a Generalized Riemann-Hilbert-Birkhoff decomposition to systematically generate and classify multi-component nonlinear integrable models. Explicit examples of the positive and negative flows of the mKdV and Chen-Lee-Liu (CLL) hierarchies and its various reductions, including Burgers hierarchy are considered. For the CLL hierarchy two classes of vacua, namely zero and non-zero constant vacuum solutions are shown to be admissible. The tau functions for soliton solutions are obtained by a dressing method and vertex operators are constructed for both types of vacua. We are able to select and classify the soliton solutions in terms of the type of vertices involved. A particular set of solitons solutions constructed by a judicious choice of vertices are shown to yield in a closed form, the multi soliton solutions for the Burgers hierarchy.

15. **Substitution principles in ideal gas dynamics and magneto gas dynamics derived as conditional symmetries**
by
M. Gorgone: University of Messina, Italy

Collaborator: F. Oliveri

Abstract: In this presentation, some classical results in ideal gas dynamics and ideal magneto gas dynamics, known as *substitution principles* [1, 2, 3, 4, 5], are considered. We observe that the transformations yielding substituted flows map solutions into solutions, so they correspond to symmetries of the governing equations. We will prove that methods of Lie group analysis of differential equations allow one to derive from infinitesimal considerations the transformations involved in the substitution principles, for both steady and unsteady equations. Remarkably, suitably using Lie group methods, we are able to obtain some generalizations. The theoretical framework we use is that of conditional symmetries, where we look for the symmetries of a system of differential equations supplemented by some side conditions.

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16. **Advances in the formal diagonalisation of systems of evolution equations**

by
Rafael Hernández Heredero: Universidad Politécnica de Madrid, Spain

Collaborators: Rafael Delgado López, Vladimir V. Sokolov

Abstract: Integrability of multi-component evolution systems

$$u^i_t = \phi^i(\mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_n), \quad i = 1, \ldots, m$$

(where $\mathbf{u} = (u^1, \ldots, u^m)$ and $\mathbf{u}_i = \partial^i \mathbf{u}/\partial x^i$) is defined, under the symmetry approach, as the existence of formal recursion $\mathbf{R}$ and symplectic $\mathbf{S}$ operators for the system. These objects are matrix pseudo-differential series satisfying the equations

$$\mathbf{R}_t = [\mathbf{\Phi}_*, \mathbf{R}],$$
$$\mathbf{S}_t + \mathbf{S}\,\mathbf{\Phi}_* + \mathbf{\Phi}_*^+\,\mathbf{S} = 0.$$

where $\mathbf{\Phi}_*$ denotes the Fréchet derivative of $\mathbf{\Phi} = (\phi^1, \ldots, \phi^m)$.

Systems that satisfy some nondegeneracy condition are known to be formally diagonalisable [1] via a gauge transformation that diagonalises the Fréchet derivative $\mathbf{\Phi}_* = \mathrm{diag}(\phi_1, \ldots, \phi_m)$, simplifying the equations for recursion and symplectic operators. For example, the equation for a recursion operator splits into m scalar equations $(R_i)_t = [\phi_i, R_i]$ and the well developed theory of integrable scalar evolution equations can be applied without further ado.

We will explore in this talk the possibility of diagonalising some degenerate systems, expanding the class of systems that can be studied under the symmetry approach, including examples of significant physical content. As an application, we will deduce explicit integrability conditions and produce a partial classification of integrable systems of the form

$$\begin{aligned} u_t &= v_1, \\ v_t &= u_3 + f(u, u_1, u_2, v, v_1). \end{aligned}$$

References

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17. **Integrability paradigm inspired by the Yang-Baxter equation**

by

Jarmo Hietarinta: University of Turku, Finland

Abstract: Consider three quantum particles (with different velocities) moving on a line. The order in which they collide depends on their initial positions, but if the Yang-Baxter equation is satisfied, the final result does not depend on the order. In this talk I will show that this basic idea can be observed in many different situations, if we use a creative interpretation of "particles" and "scattering". The canonical example leading to the Yang-Baxter equation has an immediate natural extension to the set theoretical case, as exemplified by Yang-Baxter maps. However, this paradigm can also be observed in the "Consistency-Around-a-Cube" concept for quad equations on a $\mathbb{Z}^2$ lattice, as well as in the three-soliton condition in Hirota's bilinear formalism.

18. **Some studies on integrable integro-differential equations involving the Hilbert operator**

by

Xing-Biao Hu: Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, PR China

Abstract: In this talk, I will first review several known integrable integro-differential equations involving the Hilbert operator, including the Benjamin–Ono (BO) equation, the sine-Hilbert equation, the Constantin–Lax–Majda equation, and others. I will then present recent new results achieved in this field.

19. **On the governing equations of membrane O surfaces**
by
Yoshiki Jikumaru: Toyo University, Tokyo, Japan

Abstract: It is known that a shell membrane in equilibrium where a constant purely normal load acts on the membrane, and where the principal curvature lines coincide with the principal stress lines, forms an integrable system called a membrane O surface [1]. In this talk, we formulate the governing equations for membrane O surfaces of the 1st and 2nd kind, which are analogues to Guichard surfaces of the 1st and 2nd kind introduced by Calapso. Furthermore, under this formulation, we show that membrane O surfaces are suitable subclasses of Demoulin's Ω surfaces, and that the Bäcklund transformation for membrane O surfaces preserves membrane O surfaces of the 1st and 2nd kind, respectively.

References

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20. **Wronskians over multidimension: From $\mathfrak{sl}(2)$ to (in)finite-dimensional polynomial homotopy Lie algebras**
by
Arthemy V. Kiselev: University of Groningen, The Netherlands

Abstract: By commuting three vector fields on the line $\mathbb{R} \ni x$ with monomial coefficients 1, $-2x$, and $-x^2$, we realise the Lie algebra $\mathfrak{sl}(2)$ in its Chevalley basis; the bracket acts on the coefficients as the Wronskian determinant. Let us extend this model to a class of polynomial homotopy Lie algebras in which the N-ary brackets are given by the Wronskian determinants over multidimension; the generalised Vandermonde determinants then express the structure constants.

§1. The alternated composition of $N = 2p$ differential operators $w_j(x)\,\partial_x^p$ of strict order p on the line $\mathbb{R} \ni x$ is again a differential operator of strict order p; its coefficient is the constant $c(p)$ times the Wronskian determinant of the coefficients $w_1, \ldots, w_N$. At $p = 1$, the $\mathfrak{sl}(2)$ case fixes $c(1) = 1$; easy is $c(2) = 2$, then $c(3) = 90$. In arXiv:2605.11137 [math.CO] (joint work with K. C. Shah), we reach the exact values

$c(p=4) = 586\,656$, $c(p=5) \approx 1.9 \cdot 10^{12}$, and $c(p=6) \approx 7.9 \cdot 10^{21}$. The positive integer sequence $c(p)$ seems to be new; to know $c(p \geqslant 15)$ is an open problem.

§2. Deform the binary Lie bracket to a formal sum of Wronskians with purely even ($N = 2p$) or arbitrary ($N \in \mathbb{N}_{\geqslant 2}$) arities, see `arXiv:2510.02145` [math.RA]. Not only does the full bracket Δ satisfy the Jacobi identity $\Delta[\Delta] = 0$ for homotopy Lie algebra, but for every pair of arities $\ell, m \geqslant 2$ the respective (ℓ, m)-term in the identity vanishes separately. Over base dimension $d = 1$, we spot an infinite sequence of *finite*-dimensional polynomial homotopy Lie algebras starting at $\mathfrak{sl}(2)$ and with the Wronskians as the brackets; all the structure constants, unless zero due to repetitions, equal ± 1 in a suitable basis.

§3. Let the base dimension $d \geqslant 1$ be arbitrary: $\mathbb{R}^d \ni (x^1, \ldots, x^d)$. We proved in `arXiv:math.RA/04110185` that the complete generalised Wronskians – involving all the derivatives up to a given differential order $k \geqslant 1$ – still satisfy the table of Jacobi identities for strong homotopy Lie algebras. The arity $N = \binom{d+k}{d}$ of such brackets grows with dimension d and order k but the steps, as $k \mapsto k+1$, grow as well: over $d > 1$ the gaps get larger and larger. In a recent work `arXiv:2511.03848` [math.RA] we prove that by allowing the multivariate Wronskians be *incomplete* in their top differential order $k > 1$, we do preserve all the SH-Lie Jacobi identities.

§4. For complete Wronskians of orders $k \geqslant 1$ over (multi)dimension $d \geqslant 1$ as the brackets, in arXiv:2605.27305 [math.RA] (joint with M. G. Ķēniņš) we exhaustively describe all the *finite*-dimensional polynomial N-ary SH-Lie algebra generalisations of $\mathfrak{sl}(2)$; we express their structure constants in terms of the multivariate Vandermonde determinants. Relaxing the finite-dimensionality assumption and taking the (Laurent-) monomials in d variables for the generators, we obtain multivariate analogues of the Witt algebra from CFT.

21. Conservation properties of mean field games equations

by

Roman Kozlov: Norwegian School of Economics, Norway

Abstract: The Mean Field Games (MFG) theory provides a mathematical framework for understanding the behavior of large populations of interacting agents, where the entire population influences each agent's behavior. The PDE approach to MFG theory is a rapidly developing field of research that has its roots in the seminal work of J.-M. Lasry and P.-L. Lions [1]. Another approach to the development of MFG theory was suggested by M. Huang, R. Malhamé, and P. Caines [2].

The system of mean field games equations consists of two partial differential equations: the Hamilton-Jacobi-Bellman equation for the value function and the forward Kolmogorov equation for the probability density. For separable Hamiltonians, this system has a variational structure: its equations are Euler-Lagrange equations for some Lagrangian functions. Therefore, one can use the Noether theorem to derive the conservation laws using variational and divergence symmetries.

The presentation considers separable, state-independent Hamiltonians in one-dimensional state space [3, 4, 5]. First, the most general form of the mean field games system is examined for symmetries and conservation laws. Then, particular cases of the system that lead to additional symmetries and conservation laws are identified.

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22. The "goldfish" equations of motion for infinitely many particles

by

François Leyvraz Universidad de Nacional Autonoma de Mexico, Cuernavaca, Mexico

Abstract: The "goldfish" equations of motion are nonlinear ordinary differential equations of second order connecting N particles on the complex plane. The motion becomes simple once the particles taken to be the zeroes of a monic polynomial, as the particles' motion then translates into free motion for the polynomial's coefficients.

For infinitely many particles, a similar system involves the theory of entire functions developed by Hadamard and others. It is shown that, if the initial values of the coordinates of the z_n satisfy $\sum_{n=1}^{\infty} |z_n(0)|^{-1-\epsilon} < \infty$ for some $\epsilon > 0$, and with an appropriate condition on the initial velocities, then the goldfish equations can trivially be extended by taking the $N \to \infty$ limit. If this condition is not satisfied, interesting complexities arise, and the dynamics of the zeroes is described by different equations. This will be discussed in greater detail.

23. Pencils of Novikov algebras of Stäckel type and soliton hierarchies

by

Krzysztof Marciniak: Linköping University, Campus Norrköping, Sweden

Collaborators: Maciej Błaszak and Błażej Szalikowski

Abstract: There exist various ways of constructing soliton hierarchies from appropriate algebraic structures. For example, in [10] the authors used loop algebras and r-matrix theory to produce compatible Poisson brackets leading to cKdV and cHD hierarchies. In [7] Frobenius algebras were applied to multi-component third-order local Poisson structures. In the article [13], the authors performed the construction of (1+1)-dimensional integrable bi-Hamiltonian systems associated with Novikov algebras. The obtained systems were multi-component generalizations of the Camassa-Holm equation [8] that can be interpreted as Euler equations on the respective centrally extended Lie algebras. A similar approach for constructing multi-component soliton hierarchies, specifically Harry Dym and Hunter-Saxton, based on Frobenius triple, has been presented in [12].

The homogeneous first-order Hamiltonian operators [1, 4], which are a special case of the Dubrovin-Novikov operators of hydrodynamic type [9], have a very natural underlying algebraic structure. The conditions for a homogeneous operator

$$\Pi^{ij} = \frac{1}{2}(b_k^{ij} + b_k^{ji})u^k \frac{d}{dx} + \frac{1}{2}b_k^{ij} u_x^k, \tag{1}$$

to be Hamiltonian are such that the b_k^{ij} are the structure constants of a Novikov algebra [4]. Moreover, these operators can be defined through Lie-Poisson structures associated with the so-called translationally invariant Lie algebras, which are in one-to-one correspondence with Novikov algebras.

In this talk we show a way of constructing evolutionary soliton hierarchies from pencils of Novikov algebras of Stäckel type. We begin by defining a special class of associative Novikov algebras, which we call Novikov algebras of Stäckel type, as they are associated with classical Stäckel metrics in Viète coordinates. We obtain sufficient conditions for pencils of these algebras so that the corresponding Dubrovin-Novikov Hamiltonian operators can be centrally extended, producing sets of pairwise compatible Poisson operators. These operators lead to coupled Korteweg-de Vries (cKdV) and coupled Harry Dym (cHD) hierarchies [1, 2, 3, 5] as well as to a triangular cKdV hierarchy and a triangular cHD hierarchy.

The content of this talk can be found in [6]

This talk is an effect of a joint work with Maciej Błaszak and Błażej Szalikowski, Adam Mickiewicz University, Poznań, Poland.

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24. Contractions of Lie Algebras and their Realizations

by

Maryna Nesterenko: Institute of Mathematics, NAS of Ukraine & Kyiv School of Economics, Ukraine

Abstract: A contraction of a Lie algebra is a limiting procedure that transforms one Lie algebraic structure into another. Such transformations play an important role in mathematical physics, providing systematic links between different physical theories, describing symmetry reductions, and arising naturally in the study of invariant differential equations.

In this talk, we will present several approaches to Lie algebra contractions and discuss criteria for their existence. We will analyze Hasse diagrams that encode contraction relations among low-dimensional and nilpotent Lie algebras over the real and complex fields. In addition, we will consider the problem of constructing contractions of realizations and limit relations between invariant models.

25. **Lattice Gel'fand-Dikii mappings and the associative Yang-Baxter equation**

by
Frank W. Nijhoff: University of Leeds, UK & Shanghai University, PR China

Collaborators: Steven King, Cheng Zhang, Da-jun Zhang, Peter van der Kamp

Abstract: The lattice Gel'fand-Dikii (GD) hierarchy was introduced in [1], and its mapping reductions, under periodic initial conditions, were derived. These families of mappings possess two Lax representations: a zero-curvature ('small') Lax pair, and a dual ('big') Lax pair. In the case of the small Lax pair a non-ultralocal classical r-matrix structure was given, which can be readily generalised to the quantum case, [2, 3]. In contrast, the r-matrix structure for the dual Lax representation remained elusive for a long time, and was given only in the special case of KdV mappings much later in [4]. The dual r-matrix has a rather curious structure, which will be elucidated in the talk, and obeys the so-called *associative Yang-Baxter equation* (a terminology coined by A. Kirillov). It will be shown that the relevant r-matrix extends to classical and quantum mappings associated with the entire GD hierarchy. We will discuss the associated algebraic structures and various generalisations. This is work partly in collaboration with Steven King, and partly with Cheng Zhang, Da-jun Zhang and Peter van der Kamp.

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26. **Title: On Symmetries of Differential Equations**

by
M.C. Nucci: University of Bologna, Italy

Abstract: This lecture is divided in two parts: one about partial differential equations and the other about ordinary differential equations. Concerning partial differential equations, classical and non classical symmetries will be briefly recalled and then heir-equations and their properties will be presented and exemplified. Concerning symmetries of ordinary differential equations, their link to conservation laws, hidden linearity and integrability will be illustrated with some examples.

27. An algebraic understanding of Hietarinta's 4×4 constant Yang-Baxter solutions

by

Pramod Padmanabhan: Department of Physics, School of Basic Sciences, Odisha, India

Abstract: In this talk, I will review one of Jarmo Hietarinta's achievements in the early 90's - classifying the constant Yang-Baxter solutions in the two dimensional case. I will then elaborate on our current understanding of his solutions, which include an algebraic derivation of his solutions [along the lines of the Jones braid group representations] and the Baxterization of his solutions leading to non-hermitian and hermitian integrable systems. The talk will be based on:
https://arxiv.org/abs/2409.05375, https://arxiv.org/abs/2503.08109,
https://arxiv.org/abs/2508.04315.

28. Exact Integration of Nonlinear PDEs via $\mathcal{C}^\infty$-Structures

by

A. J. Pan-Collantes: Department of Mathematics, Universidad de Cádiz, Spain

Abstract: We address the problem of finding explicit solutions to nonlinear PDEs by combining the method of differential constraints with the theory of $\mathcal{C}^\infty$-structures. The method of differential constraints [1, 2, 6] reduces a PDE to an overdetermined system whose compatibility can be studied via its Vessiot distribution. Recent applications of compatible differential constraints to nonlinear evolution equations [7] illustrate the scope of this strategy. Moreover, when this distribution is involutive on an open set and the rank equals the number of independent variables, it can be integrated by quadratures using a solvable structure [3, 4], though solvable structures remain difficult to construct in practice.

We propose replacing them by $\mathcal{C}^\infty$-structures of distributions [5], a generalization, with relaxed bracket conditions, that is directly analogous to the extension of Lie point symmetries to $\mathcal{C}^\infty$-symmetries (or λ-symmetries) in the ODE setting. Just as λ-symmetries of an ODE enable integration even when no Lie point symmetry exists, a $\mathcal{C}^\infty$-structure of a Vessiot distribution provides a systematic sequence of completely integrable Pfaffian equations whose solution yields the exact integral manifolds of the distribution, and hence explicit solutions of the original PDE.

We illustrate the method with explicit examples on nonlinear evolution equations where classical solvable structures are unavailable.

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29. **Classical symmetry analysis of differential equations revisited**

by

Roman Popovych: Silesian University in Opava, Czech Republic, & Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

Abstract: We discuss the rethinking of the foundations of classical symmetric analysis of differential equations that has been carried out over the past two decades. This mainly includes point transformations of differential equations and structures constituted by such transformations.

Fist we concentrate on point symmetries of differential equations, distinguishing continuous and discrete ones, and the methods of their computation. Common misconceptions in interpreting point symmetries and errors in their computation are analyzed. The main illustrative example is given by the (1+1)-dimensional linear heat equation. We derive a nice representation for its point symmetry transformations and properly interpret them. This allows us to prove that the pseudogroup of these transformations has exactly two connected components. That is, the heat equation admits a single independent discrete point symmetry, which can be chosen to be alternating the sign of the dependent variable. The developed approach to point-symmetry groups whose elements have components that are linear fractional in some variables is extended to many other linear and nonlinear differential equations. In particular, the Burgers equation admits no discrete point symmetries.

We also briefly discuss point transformations in classes of differential equations in this context, starting from the notion of such classes and various natural structures

constituted by point transformations within them. These structures in particular include equivalence groupoids and many kinds of equivalence groups (usual, generalized, extended, extended generalized, effective generalized, effective extended generalized, gauge and insignificant ones). A proper formulation of group classification problems in classes of differential equations is presented as well.

30. **The differential geometry of the (modified$^{(2)}$) Korteweg-de Vries equation and associated Miura transformations**

by

Wolfgang K. Schief: School of Mathematics and Statistics, The University of New South Wales, Sydney, Australia

Abstract: We present a framework in three-dimensional Minkowski space $\mathbb{R}^{1,2}$ which unifies the extended Dym, KdV, modified KdV and modified modified KdV equations via parallel, offset and midsurfaces. Each equation governs a class of surfaces, the members of which are foliated by geodesics of certain properties. These classes of surfaces are linked by reciprocal and Miura-type transformations. In particular, we obtain a novel geometric interpretation of the classical Miura transformation linking the KdV and mKdV equations. In total, there exist ten classes which may be associated both combinatorially and literally with the 4 vertices and 6 midpoints of the edges of a (moving) tetrahedron.

31. **Refined Painlevé/gauge theory correspondence**

by

Anton Shchechkin: SISSA, INFN, and IGAP, Trieste, Italy

Collaborators: Giulio Bonelli, Alessandro Tanzini

Abstract: We present bilinear tau forms of the quantum Painlevé equations and describe their solutions near critical points in terms of supersymmetric gauge theory partition functions.

We seek these solutions as Zak transforms of certain (asymptotic) series around critical points of the (quantum) Painlevé equations, by quantizing the ansatz of seminal paper [1] and subsequent [2]. We compute several leading terms of these series and fix the arising freedom of the bilinear tau forms from the Hamiltonian forms of the quantum Painlevé equations. We then identify the obtained series with the (asymptotic) expansions of partition functions of four-dimensional $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theories with appropriate Ω-background. In this way, we refine the Painlevé/gauge theory correspondence of [1], [2].

We mainly focus on the asymptotic expansions around the irregular critical point $t = \infty$, which correspond to the strong-coupling regime of the gauge theories, including Argyres-Douglas points. On the gauge-theory side, we compute these expansions

using the holomorphic anomaly equations. We also identify them with expansions of irregular conformal blocks, thereby clarifying the corresponding AGT relation.

This talk is based on [3] and its companion paper [4].

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32. **Matrix Darboux transformations for discrete integrable systems**
by
Ying Shi: Zhejiang University of Science and Technology, PR China

Abstract: This talk is devoted to matrix Darboux transformations for discrete integrable systems. It is motivated by earlier work on reductions of the Hirota-Miwa equation and Darboux transformations for certain members of the ABS list. We consider Lax pairs in matrix form for discrete integrable systems, while the underlying potentials remain scalar. Within this framework, we construct Darboux transformations at the matrix level and investigate their algebraic properties. The matrix formulation provides a systematic way for generating new solutions from known ones and enables an efficient description of iterated transformations.

33. **Geometric Lagrangian one-forms and Hitchin systems**
by
Anup Anand Singh: Loughborough University, UK

Collaborators: V. Caudrelier, M. Dell'Atti, D. Harland, B. Vicedo

Abstract: The theory of Lagrangian multiforms provides a variational description of integrable hierarchies using a generalised variational principle applied to an appropriate generalisation of a classical action. The case of Lagrangian one-forms covers finite-dimensional integrable systems.

In the first part of the talk, I will present an overview of geometric Lagrangian one-forms: a novel variational framework formulated in phase space. I will also briefly discuss its connection with the more traditional Hamiltonian approach to integrability by showing how the closure relation for Lagrangian one-forms serves as the variational analogue of the Poisson involutivity of Hamiltonians.

The second part of the talk concerns the construction of geometric Lagrangian one-forms for Hitchin systems, a large class of integrable systems of algebro-geometric origin. I will show how adapting Hitchin's construction to the variational setting of Lagrangian multiforms produces a multiform version of the action of the 3d holomorphic-topological BF theory with defects. Moving to a holomorphic local trivialisation of principal G-bundles yields a simple 1d action which unifies several well-known integrable hierarchies — including those of rational and elliptic Gaudin models and a spin generalisation of the elliptic spin Calogero-Moser model — within a single variational framework.

The talk is based on joint works with V. Caudrelier, M. Dell'Atti, D. Harland, and B. Vicedo.

34. **Non-autonomous finite-dimensional restrictions of soliton hierarchies and Painlevé type systems**
by
Błażej Szablikowski: *Adam Mickiewicz University, Poland*

Collaborators: Maciej Błaszak and Krzysztof Marciniak

Abstract: I will present results of joint work with **Maciej Błaszak**(UAM, Poznań) and **Krzysztof Marciniak** (Linköping University).

Since the classical works of Novikov et al., there has been a tremendous amount of research devoted to connections between soliton hierarchies and their integrable finite-dimensional reductions, which was mainly focused on stationary flows. Recently, we have revisited this idea in a novel and systematic way [1, 2, 3]. We investigated not only stationary flows but also the so-called stationary systems, by which we mean a stationary flow together with all lower flows from the hierarchy, that is, finite-dimensional systems of evolutionary equations. As a result, we were able to show that, in the case of particular soliton hierarchies, the related stationary systems can be represented as classical separable Stäckel systems.

Here, we generalize the above concept of stationary systems to the so-called non-autonomous restrictions of soliton hierarchies. These restrictions are defined through invariant time-dependent constraints that are appropriate deformations of stationary flows through compositions of the so-called master symmetries and lower flows, an idea based on [4]. It turns out that this class of time-dependent restrictions of soliton hierarchies, at least in particular cases, is represented by non-autonomous Hamiltonian finite-dimensional dynamical systems of Painlevé type. Let us emphasize that the original Painlevé equations are non-autonomous nonlinear ODEs that, at the beginning of the 20th century, led to the definition of new transcendental

special functions. I will illustrate our theory by considering the Korteweg–de Vries (KdV) hierarchy and its coupled generalizations, in particular the Dispersive Water Waves (DWW) hierarchy.

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35. **Time-dependent Trapped Plasma Nonlinear Dynamics**

by

Ronaldo Thibes: Departamento de Ciências Exatas e Naturais, Universidade Estadual do Sudoeste da Bahia, Itapetinga BA, Brazil

Abstract: We investigate the nonlinear dynamics of a single-component plasma confined in a time-dependent harmonic trap. The system is described as a fluid in a hydrodynamic framework, incorporating electrostatic and thermal effects. By applying a time-dependent variational method, with a Gaussian ansatz for the number density distribution, we reduce the plasma complexity to that of a system of ordinary differential equations and follow up with a consistent analysis of the corresponding Noether symmetries in three distinct regimes: electrostatic, thermal, and combined effects. For each studied case, proper conserved quantities are identified, including a generalized Ermakov-Lewis invariant. The plasma dynamics is captured by a reduced set of ordinary differential equations derived from variational principles, in which the Lagrangian formulation naturally incorporates electrostatic self-interactions and thermal pressure effects, while preserving relevant dynamical features and offering a powerful framework for identifying symmetries, conserved quantities, and invariants. The obtained conserved quantities capture an interplay between the internal plasma dynamics and the time modulation of the trap in an integrable way, resulting in a rigorous restriction for the system evolution in all considered regimes. The presence of the invariants highlight the fundamental relationship between the symmetry, conservation laws and integrability present in the single-component plasma dynamics.

36. **On application of nonpoint symmetry reduction method to nonlinear evolutionary and wave type equations**

by
Ivan Tsyfra: AGH University of Krakow, Poland

Abstract: We study the symmetry reduction of partial differential equations with two independent variables. We construct an ansatz for the dependent variable or its derivatives that reduces a scalar partial differential equation to a system of ordinary differential equations. One can obtain the ansatz for the dependent variable by solving an ordinary differential equation admitting operators of generalized symmetry. It is worth noting that, in the case of an evolutionary equation, the number of differential equations in the reduced system equals the number of unknown functions. At the same time, in the case of non-evolutionary equations, the number of ordinary differential equations in the reduced system may be less than the number of unknown functions. This enables us to obtain solutions that depend on arbitrary functions, as illustrated in the report by examples of nonlinear wave-type equations. We also show that this method can be applied to a system of two evolutionary equations related to the KdV equation.

37. **Symplectic operators and Lagrangian multiforms for bi-Hamiltonian systems**

by
Mats Vermeeren: Loughborough University, UK

Collaborator: Pierandrea Vergallo

Abstract: This talk presents some recent insights into the Lagrangian structure of bi-Hamiltonian systems. It is well-known that for many integrable Hamiltonian PDEs (e.g. KdV equation) passing to a potential variable allows one to formulate a variational principle. We show that such a potential variable is fundamentally connected to the Hamiltonian operator. The transformation to potential variables turns a compatible pair of Hamiltonian operators into a compatible pair of symplectic operators. Each of these symplectic operators can be used to construct a Lagrangian multiform - a structure that describes an integrable hierarchy in a single variational principle.

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38. **Bi-Hamiltonian systems from homogeneous operators**
by
Raffaele Vitolo: Università del Salento, Italy

Collaborators: P. Lorenzoni, S. Opanasenko

Abstract: Many 'famous' integrable systems (KdV, AKNS, dispersive water waves etc.) have a bi-Hamiltonian pair of the following form: $A_1 = P_1 + R_k$ and $A_2 = P_2$, where P_1, P_2 are homogeneous first-order Hamiltonian operators and R_k is a homogeneous Hamiltonian operator of degree (order) k. The Hamiltonian property of P_1, P_2 and their compatibility were given an explicit analytic form and geometric interpretation long ago (Dubrovin, Novikov, Ferapontov, Mokhov). The Hamiltonian property of R_k was studied in the past (Doyle, Potemin; $k = 2, 3$) and recently revisited with interesting results. In this talk, we illustrate the analytic form and some preliminary geometric interpretation of the compatibility conditions between P_i and R_k, $k = 2, 3$. See the recent papers https://arxiv.org/abs/2602.14739, https://arxiv.org/abs/2407.17189, https://arxiv.org/abs/2311.13932.
Joint work with P. Lorenzoni and S. Opanasenko

39. **Soliton-like solutions supported by refined hydrodynamic-type model of an elastic medium with soft inclusions**
by
Vsevolod Vladimirov: AGH University of Science and Technology, Poland

Abstract: A modified model of nonlinear elastic medium containing sharp inhomogeneities is considered. The modification consists in introducing into the dynamic equation of state those terms that were discarded in the previously considered models. The main purpose of the ongoing research is to analyze the existence, stability and dynamic properties of soliton-like solutions. It is shown that, under certain restrictions, the modified model describes the solitary waves of compression and rarefaction, that is, among its solutions there are the same wave structures as in the previously considered model. However, this is where the coincidence of the properties ends, since it is strictly proved within the previously considered model, that only solitary waves of rarefaction are spectrally stable.

The results of current research show that the solitary waves of rarefaction supported by the modified model are stable. At the same time, previously unstable solitary waves of compression acquire the stability due to the incorporation of the higher order terms of the asymptotic expansion into the dynamic equation of state.

40. **A peculiar nonautonomous form of the Lyness mapping**
by
Ralph Willox: University of Tokyo, Japan

Collaborators: Basil Grammaticos, Alfred Ramani

Abstract: We examine possible integrable deautonomisations of the family of mappings commonly known as 'the Lyness mapping' [1–3] —N^{th}-order integrable discrete systems ($N \geq 2$) that can be generated from a one-dimensional reduction of the Hirota-Miwa equation— based on the structure of their singularities. We show that when the mappings are given in their standard form,

$$x_{n+N}x_n = a + x_{n+1} + x_{n+2} + \cdots + x_{n+N-1},$$

only the N = 2 case has a natural deautonomisation, whereas the mappings in their 'derivative form',

$$x_{n+N}(1 + x_n) = x_{n+1}(1 + x_{n+N+1}),$$

have natural, integrable, deautonomisations for all $N \geq 3$. However, the deautonomisation of the derivative form in the $N = 2$ case turns out to possess a feature we have never met before: the secular dependence on n in the coefficients of the mapping enters through two different exponential terms instead of just a single one. As a consequence, as a special subcase, the same mapping can also have an integrable nonautonomous form in which the n dependence appears in an additive fashion in the coefficients instead of the generic multiplicative dependence.

This talk is based on a paper together with Basil Grammaticos and Alfred Ramani [4].

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41. **Novel existence conjectures and unified constraints for higher-order peakons/pseudo peakons in generalized J-K-CH equations**

by

Ruoxia Yao: Shaanxi Normal University, PR China

Abstract: In this talk, we will focuse on investigating peakon and pseudo-peakon waves for the generalized higher-order Camassa-Holm (CH) equations, J-K-CH. By utilizing the weak solution of the generalized J-K-CH equations and leveraging properties of the signum function, generalized/distribution functions, we establish a theoretical framework to construct higher-order peakon and pseudo- peakon waves and give some conjectures related to the forms and properties of the higher-order peakon waves and pseudo-peakon waves. Then, analyzing the unified constraint conditions, we derive some results: for J-1-CH equation ($\forall J \geq 4$), a 5th-order pseudo-peakon; for J-2-CH equation ($\forall J \geq 6$), a 7th- order pseudo-peakon; for J-3-CH equation ($\forall J \geq 8$), a 9th-order pseudo- peakon; and for the generalized J-K-CH equation ($\forall J \geq 2(K+1)$), a $(2K+3)$th-order pseudo-peakon. Also, more higher-order pseudo-peakons are given by analyzing the continuity and the singularity conditions respectively.

42. **The GLM equation and direct linearisation structure related to the Lamé function**

by

Da-jun Zhang: Shanghai University, PR China

Collaborators: Xing Li, Ying-ying Sun

Abstract: We will establish an elliptic direct linearization (DL) scheme for the KP equation. The scheme consists of an integral equation involving the Lamé function. A formula for elliptic soliton solutions is confirmed in this scheme by checking Lax pair of the KP. Based on analysis of real-valuedness of the Weierstrass functions, we are able to construct a GLM equation for elliptic solitons for the KP equation. By utilizing elliptic Nth roots of unity and reductions, the elliptic DL schemes, GLM equations and nonsingular real solutions can be obtained for the KdV equation and Boussinesq equation. The paper is based on a joint work with Xing Li and Ying-ying Sun: Nonlinearity, 38 (2025) No.105024 (arxiv:2501.06476).

43. **Direct linearization and bilinear structure of a fourth order lattice Gel'fand-Dikii equation**

by

Songlin Zhao: Zhejiang University of Technology, PR China

Collaborators: Guesh Yfter Tela, Han Wang, Da-jun Zhang

Abstract: Utilizing the direct linearization approach, we present a fourth order lattice Gel'fand-Dikii (lattice GD-4) equation. This equation is related to a quartic discrete dispersion relation and can be viewed as higher-order member of the lattice Boussinesq equation. The resulting equation is in five-component form, and it is multi-dimensionally consistent by introducing extra equation. Lax integrability is discussed both by direct linearization scheme and also through multidimensional consistent property. Bilinear form and solution in Casoratian of this equation are presented. Based on the obtained soliton solutions, we extend this equation by introducing a parameter δ. This δ-extended lattice GD-4 equation is still consistent around the cube, and its bilinear form together with Casoratian solutions are provided. These works are joint with Guesh Yfter Tela, Han Wang and Da-jun Zhang.

44. **The integrable Volterra system and its super-integrability**
by
Federico Zullo: DICATAM, Università degli Studi di Brescia, Italy & INFN, Italy

Collaborator: O. Ragnisco

Abstract: We present the results contained in three recent works [1]-[3], all published in OCNMP, where it has been shown that the integrable version of the N-species Volterra model, introduced by Vito Volterra in 1937, is indeed maximally super-integrable. This superintegrability property applies as well to the case of infinitely many competing species, either countable or uncountable. It is shown that the model can be reduced to a Hamiltonian system with only one degree of freedom. The particular form the Hamiltonian assumes depend on the parameters of the model. We give different examples by expliciting also the properties of the corresponding dynamics.

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Open Problems Session

1. **Title:** To be announced
by
Stephen Anco: Brock University, Canada

Abstract:

2. **Open problem: discretize Gauss-Codazzi equations**
by
Robert Conte: LRC MESO, Centre Borelli, France & The University of Hong Kong

Abstract: The moving frame of surfaces in the usual space $\mathbb{R}^3$ (Bobenko 1994)

$$\mathbb{U} = \begin{pmatrix} (1/4)u_z & -Qe^{-u/2} \\ (1/2)(H+c)e^{u/2} & -(1/4)u_z \end{pmatrix}, \mathbb{V} = \begin{pmatrix} -(1/4)u_{\bar{z}} & -(1/2)(H-c)e^{u/2} \\ \overline{Q}e^{-u/2} & (1/4)u_{\bar{z}} \end{pmatrix}, \qquad (1)$$

generates by the zero-curvature condition

$$[\partial_z - \mathbb{U}, \partial_{\bar{z}} - \mathbb{V}] = \mathbb{U}_{\bar{z}} - \mathbb{V}_z + [\mathbb{U}, \mathbb{V}[= 0, \qquad (2)$$

three PDEs in four fields (u and H real, Q and $\overline{Q}$ complex conjugate), the Gauss-Codazzi equations

$$\begin{cases} u_{z\bar{z}} + \frac{1}{2}H^2 e^u - 2|Q|^2 e^{-u} = 0 \text{ (Gauss)}, \\ Q_{\bar{z}} - \frac{1}{2}H_z e^u = 0,\ \overline{Q}_z - \frac{1}{2}H_{\bar{z}} e^u = 0 \text{ (Codazzi)}. \end{cases} \qquad (3)$$

Open problem. Discretize (d- or q-) either the linear system (1) or the nonlinear system (3) and, of course, require their continuum limit to be the continuous equations.

There exist many partial results (Bobenko, Konopelchenko, Nijhoff, Schief, ...), but not this general one.

Motivation. Since the moving frame of Bonnet surfaces is equivalent to the best matrix Lax pair of continuous $\mathrm{P_{VI}}$ (RC 2017), a by-product should be the best discrete matrix Lax pair of possibly the best discrete $\mathrm{P_{VI}}$.

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3. **Discrete Painlevé Equaitons: what next?**
by
Anton Dzhamay: Beijing Institute of Mathematical Sciences and Applications, PR China

Abstract: The general theory of discrete Painlevé equations was laid out in the seminal paper of H. Sakai [Comm. Math. Physics, 2001]. This paper gives a full classification of possible configuration spaces, now called the Sakai surfaces (more accurately, surface families) of discrete Painlevé dynamics. The symemtry groups of these spaces are certain affine Weyl groups that also generate the discrete Painlevé dynamics, which corresponds to translation or quasi-translation elements in these groups. This theory by now is well-established. The question I want to address is how can it be extended and generalized.

There are two directions one can look at this: specification and generalization. The first includes discrete Painlevé equations with constraints, their symmetry groups, classification and significance for the aplications. The second asks for the meaningful generalization of the Sakai geometric approach for higher order or, equivalently, higher-dimensional systems.

4. **Is there a dressing transformation method for Yang-Mills theory?**
by
Luiz Agostinho Ferreira: Instituto de Física de São Carlos; IFSC/USP; Universidade de São Paulo, São Carlos-SP, Brazil

Abstract: It has recently been shown that the dynamics of classical, non-supersymmetric, Yang-Mills theories coupled to spin-1/2 and spin-0 matter fields, in (3+1)-dimensional Minkowski space-time M, is equivalent to the zero curvature condition [1]

$$\delta\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$$

where $\mathcal{A}$ is a one-form connection on the generalized loop space

$$\mathcal{L}^{(2)} = \left\{ f : S^2 \to M \mid \text{north pole} \to x_R \right\}$$

It then follows that the Yang-Mills theories possess an infinite dimensional hidden symmetry group $\widehat{G}$, given by the gauge transformations

$$\mathcal{A} \to \widehat{g}\,\mathcal{A}\,\widehat{g}^{-1} + \delta\widehat{g}\,\widehat{g}^{-1}$$

where the group elements $\widehat{g}$ are holonomies of one-form connections $\mathfrak{a}$ on the loop space

$$\mathcal{L}^{(1)} = \left\{ f : S^1 \to M \mid \text{north pole} \to x_R \right\}$$

The question is: Could we devise a Riemann-Hilbert like problem that could lead us to develop some sort of dressing transformation method for the construction of solutions of Yang-Mills theory?

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5. **Periodic solutions for integrable hierarchies**
by
Jose Francisco Gomes: IFT-Unesp, Brazil

Abstract: The algebraic structure of integrable hierarchies relies on affine Lie algebras and zero curvature representations, providing a systematic dressing method for constructing soliton solutions on both zero and constant non-zero vacua. Explicitly for the prototype of the mKdV hierarchy, a third kind of solution are known to exist. These are the periodic solutions written in terms of Jacobi theta functions. Expressing quasi-periodic solutions via vertex operators remains a challenging open problem for integrable hierarchies.

6. **Open search problems**
by
Jarmo Hietarinta: University of Turku, Finland

Abstract: The following problems relate to searching for integrable systems. The assignments are easy to give but solving them requires massive use of symbolic computation.

(a) Find all 2D flat space Hamiltonians with a second invariant **cubic in momenta**. Some interesting results of this type have been found by Fokas, Holt, and Inozemtsev, but the complete solution is still open.

(b) Solve the Yang-Baxter equation in the **"scalene" version** $A_{12}B_{13}C_{23} = C_{23}B_{13}A_{12}$. This is difficult even when restricted to the 8 vertex case.

(c) Find all integrable quad equations **without the tetrahedron property** or prove that there are only trivial ones (e.g., $u_{12}u = u_1u_2$).

7. **The alternated composition of $N = 2p$ differential operators $w_j(x)\,\partial_x^p$ yields the weights' Wronskian with which constant ?**
by
Arthemy V. Kiselev: Bernoulli Institute for Mathematics, University of Groningen, The Netherlands

Abstract: We study strongly homotopy deformations of Lie algebras realised by vector fields on the (complex) line, e.g., polynomial realisations of finite-dimensional Lie algebras like $\mathfrak{sl}(2)$ or Laurent polynomial realisations of infinite-dimensional Lie algebras like the Witt algebra of holomorphic vector fields on $\mathbb{C}$. The Lie bracket of two vector fields is a vector field; its coefficient is the Wronskian of the two old coefficients. In the course of homotopy deformation, vector fields extend to higher-order differential operators, and the binary bracket acquires the tail of multilinear antisymmetric brackets of many arguments. The alternated composition of $N = 2p$ differential operators $w_j(x)\,\partial_x^p$ of strict order $p \in \mathbb{N}_{\geqslant 1}$ on the line $\mathbb{R} \ni x$ is again a differential operator of strict order p; its coefficient is the constant $\boldsymbol{c}(p)$, depending only on the arity N, times the Wronskian determinant of the originally taken coefficients $w_1, \ldots, w_N$. The case $p = 1$ of the Lie bracket for two vector fields fixes $\boldsymbol{c}(1) = 1$. When $p = 2$, finding $\boldsymbol{c}(2) = 2$ is easy; next, $\boldsymbol{c}(3) = 90$. The problem is to know $\boldsymbol{c}(p \geqslant 4)$.

In arXiv:2605.11137 [math.CO] (joint work in progress with K. C. Shah) we express the formula of $\boldsymbol{c}(p)$ in terms of the sum with signs over the subset ($\subsetneq \mathbb{S}_{2p}$) of 'late-growing' permutations, thus reaching the exact values $\boldsymbol{c}(p = 4) = 586\,656$, $\boldsymbol{c}(p = 5) = 1.9151\ldots \cdot 10^{12}$, $\ldots$, $\boldsymbol{c}(p = 13) = 8.3963\ldots \cdot 10^{197}$; the integer sequence $\boldsymbol{c}(p)$ seems to be new.

Problem. Is it true that the constants never vanish, $\boldsymbol{c}(p) \neq 0$, and $\boldsymbol{c}(p) > 0$ for all $p \in \mathbb{N}_{\geqslant 1}$?

- What is the (asymptotic) growth law for the integer sequence $\boldsymbol{c}(p)$ as $p \to +\infty$? To facilitate the answer, we ask:
- What is the law of prime decomposition for $\boldsymbol{c}(p)$?

We observe that each known value $\boldsymbol{c}(p)$ is the product of a nucleous of small primes ($\leqslant p$) in high powers, times a prime from the range $p \ldots 2p$, times a few very big primes.

- What is the law of growth for the *least prime* $\geqslant N$, and for the *largest prime* in the decomposition of $\boldsymbol{c}(p)$ as $p \to +\infty$?

The caveat $\boldsymbol{c}(p) \not\equiv 1$ could mean the following in Theoretical Physics: whenever one attempts an L_∞-deformation of the Witt algebra of holomorphic vector fields on the complex plane $\mathbb{C}$ or of a finite-dimensional Lie algebra realised by vector fields, and whenever the deformation itself is realised by higher-order differential operators, the *norms* of the expansion coefficients $w_j(x)$ must cumulatively decay faster than the pre-factor $\boldsymbol{c}(p)$. Otherwise, the deformation becomes formal, as the coefficients blow up under iteration of the bracket. In summary, does any "physical sense" predict the decay of the coefficients within the homotopy deformation tails, or does it make the L_∞-deformations of the Witt algebra in CFT taboo?

8. Geometry of integrable systems defined by curves other than hyperelliptic ones

by

Krzysztof Marciniak: Linköping University, Campus Norrköping, Sweden

Abstract: In this short presentation I will address the issue of finding relevant geometric framework for integrable systems generated by separation curves other than hyper-elliptic ones.

Every smooth n-parameter curve in the (λ, μ) plane

$$\phi(\lambda, \mu, h_1, \ldots, h_n) = 0 \tag{1}$$

leads to a Liouville integrable and separable finite-dimensional Hamiltonian system with n Hamiltonians in involution

$$h_i = h_i(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n), \quad i = 1, \ldots, n, \qquad \{h_i, h_j\} = 0 \tag{2}$$

The vast majority of research focuses on curves of hyper-elliptic type:

$$\sigma(\lambda) + \sum_{i=1}^{n} h_i \lambda^{\gamma_i} = f(\lambda)\mu^2 \tag{3}$$

where $\gamma_i \in \mathbb{Z}$, σ and f are polynomials, that generate quadratic in momenta separable integrable systems for which there exist a natural geometric interpretation. The corresponding Hamiltonians constitute an integrable system in a pseudo-riemannian space, and a subalgebra of Killing tensors that spans the separation web exists. A sub-class with flat metrics is recognized, and flat coordinates are found. Multi-hamiltonian structure with corresponding Miura-type maps and Lax formulation are well understood.

Virtually nothing is known, however, if we leave this class of curves and pass to higher order algebraic case, apart from some specific cases (see for example the curve for the stationary Bousinessq system in [3] that is third order in μ). No natural geometry connected to such systems is known. Is there any way of relating a pseudo-riemannian metric to such systems? Is there any canonical (albeit non pointwise) transformation of phase-space variables that turn these systems into systems with some of the Hamiltonians quadratic in momenta? Are there any Killing tensors related to such systems? How to find Lax formulation for such systems?

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9. **Do Integrable Systems have a future?**

by

Frank W. Nijhoff: University of Leeds, UK & Shanghai University, PR China

Abstract: When it comes to open problems in the area of integrable systems the suggestion is sometimes given that the subject is at the end of its life and that virtually everything has been solved. I am of the opposite conviction, namely that, in spite of the enormous amount of progress in many areas, and the impressive impact the subject has had on the development of modern mathematics (e.g. quantum groups, knot theory, inverse analysis, difference geometry, algebraic geometry, cluster algebras, numerical analysis and mathematical physics in general) we are still in a sense 'scratching the surface' and profound questions remain unaddressed, even ignored. Solving these questions may not only fill open gaps in the area, but may also lead to deeper understanding as well as perhaps open new areas of mathematics.

Without giving away clues here already (I don't want to destroy the suspense by revealing the 'who's dunnits' in advance) I will mention a few concrete problems in three areas:

(a) Painlevé type systems and reductions;

(b) integrable lattice equations and their solutions;

(c) integrable quantum systems and Lagrangian structures.

Furthermore, if time allows, I will try and develop ideas on where I see the subject of integrable systems, and its potential impact on foundational mathematical physics, could, or indeed should, be going, and the interplay between classical & quantum as well as the corresponding notions of space-time.

10. **Open problems of symmetry analysis of differential equations**

by

Roman Popovych: Silesian University in Opava, Czech Republic, & Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

Abstract: Despite being a classical and well-established field of mathematics, symmetry analysis of differential equations has a number of various long-staying open problems of different levels, ranging from theoretical problems on its foundations to very particular problems for specific systems of differential equations or their classes that arise in real-world applications. We discuss two sets of such problems, theoretical and particular ones. The first set includes the creation of the theory of reduction modules (i.e., nonclassical symmetries) of general systems of differential equations starting from a proper definition of these objects as well as the problems on existence and uniqueness of effective generalized equivalence groups of classes of differential equations and on proper gauging of potential systems. As particular open problems we mention the exhaustive classification of reduction modules, generalized symmetries and local conservation laws of nonlinear Klein–Gordon equations of the form $u_{xy} = F(u)$, the complete description of the space of conservation laws of the Euler equations and the construction of the equivalence groupoid of the class of shallow water equations with variable bottom topography.

11. **Some open problems in Hamiltonian PDEs**
by
Raffaele Vitolo: Università del Salento, Italy

Abstract: I review some open problems that appeared in my research activity on the Hamiltonian formalism for PDEs in the last 10 years

12. **On some crucial (but potentially difficult...) open problems to do with higher order birational mappings**
by
Ralph Willox: University of Tokyo, Japan

Abstract: During the **Open Problems Session**, I will explain the precise mathematical context in which the following open problems concerning higher order birational mappings arise:

- Can one construct examples of *non-autonomous* higher order *integrable* maps with coefficients that contain multiple exponential functions (of the variable n that counts the number of iterates), and not just two as in the case of the Lyness mapping (the subject of my lecture; see also *arXiv:2603.24871 [nlin.SI]*)?
- Do there exist higher order birational mappings (not necessarily integrable) which have a non-autonomous extension that preserves their dynamical degree, but for which that non-autonomous version cannot be obtained solely based on the singularity structure of the mapping (cf. *arXiv:2602.04147 [nlin.SI]*)
- Does there exist an example of a *non-integrable* birational map which has a non-autonomous extension that preserves its dynamical degree but that contains coefficients the asymptotic behaviour of which does not reflect that dynamical degree (cf. *arXiv:2306.01372 [nlin.SI]*) ?

13. **Algebraic structures of the vertex operators of elliptic solitons**
by
Da-jun Zhang: Shanghai University, PR China

Abstract: We take the KdV equation as an example to explain the problem. For the classical solitons of the KdV equation, it is well known that the τ function can be generated by a vertex operator that corresponds to $A_1^{(1)}$. Here by "elliptic solitons" we mean the soliton solutions living on an elliptic background and being characterized by the Lamé function (instead of the usual exponential functions). Recently we found how the elliptic soiltons of the KP, KdV and Boussinesq equation are formulated in bilinear method, and their τ functions can be generated by the "elliptic type" vertex operators [J. Nonlinear Science, 32 (2022) No.70]. However, so far it is not known how such "elliptic type" vertex operators can be connected to any affine Lie algebras. I will introduce the context of this problem. A more general problem is how such elliptic solitons can be formulated in Sato's theory.

Student Session

1. **Exact traveling waves for a damped wave equation**
by
A. Castejón: University of Cadiz, Cadiz, Spain

Collaborator: A. Ruiz

Abstract: In this talk, we address the determination of exact traveling wave solutions for a strongly damped wave equation with a combined power-type nonlinearity given by

$$u_{tt} - u_{xx} + \gamma u^{m-1} u_t = \mu u - \sigma u^q, \quad q \neq 1.$$

Strongly damped wave equations have attracted much attention due to their applications in different physical fields as modeling viscoelastic fluid flows [1] and heat conduction [2]. This equation serves as a generalized damped Landau-Ginzburg-Higgs model, describing superconductivity and drift cyclotron waves in centrifugally inhomogeneous plasmas [3]. By applying the traveling wave transformation $y = x - ct$ and redefining the parameters, the equation reduces to the second-order ordinary differential equation

$$w_2 - A_1 w^{m-1} w_1 = A_2 w + A_3 w^q. \tag{1}$$

This second-order equation is studied by means of the λ-symmetry-based integration method. The notion of λ-symmetry was introduced in [4] as a generalization of the

classical concept of Lie symmetry [5]. Since there, λ-symmetries have been applied in the literature to obtain exact solutions of ODEs even if they do not admit enough symmetries to guarantee integration by quadratures [6, 7].

Three different families of equations of the form (1) admitting λ-symmetries are identified, and the λ-symmetry-based integration method is applied to determine solutions to such equations. Two of them are solved for specific values of the parameter m, whereas the other one is fully integrated. Once the families have been integrated, the corresponding traveling waves are represented for specific parameter values to illustrate particular solutions, such as kink or antikink waves, oscillatory waves, or dipole solitons.

Finally, the results obtained via the λ-symmetry method are compared with existing approaches appearing in the literature, such as the classical Lie symmetry procedure [5, 8] and the point adjoint-symmetry method [8].

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2. **General parametric solution of a third-order system of ODEs arising in classification of integrable systems**
by
B. de la Flor: University of Cadiz, Cadiz, Spain

Collaborator: A. Ruiz, C. Muriel

Abstract: In this talk, we present the general solution to a system of two third–order ordinary differential equations (ODEs) arising from the classification problem of diagonalizable hydrodynamic chains. The Lie symmetry algebra of the system is proved to be four–dimensional and hence complete integrability by quadratures is not assured. Nevertheless, the Lie algebra permits a reduction to a non–linear, non–autonomous system of two first–order equations. Determination of symmetries of the reduced system seems to be difficult. In fact, the software Maple does not provide any symmetries for it. However, the reduced system can be transformed into a second–order ODE that admits exactly one Lie point symmetry. This underlying symmetry enables us to decouple the reduced system into an Abel equation and an auxiliary first–order ODE, the latter being solvable by quadratures once the Abel equation is integrated. The integration of the Abel equation is achieved by determining a rational first integral. Whereas theoretically the first integral is enough to solve the auxiliary equation in practice this cannot be effectively achieved due to the rather involved expression of the first integral. Alternatively, the first integral can be used to define an irreducible algebraic curve of genus zero. Exploiting this geometric property, we construct a parametric representation of the solution, allowing us to explicitly integrate the auxiliary equation. Consequently, we obtain the general solution of the reduced system and, after four subsequent quadratures, recover the exact general solution of the original third–order system in parametric form

3. **Lie symmetries, reductions and exact solutions of a (2+1)-dimensional Boussinesq equation with arbitrary nonlinearity**

by
Şeyma Gönül: Department of Mathematics, Istanbul Technical University, Istanbul, Türkiye

Collaborator: Cihangir Özemir

Abstract: In this talk, we investigate Lie symmetries of the $(2+1)$-dimensional Boussinesq equation, which was proposed to model the propagation of gravity waves on the water surface, with particular emphasis on the head-on collision of oblique waves. By considering the nonlinearity as an arbitrary function $f(u)$ rather than restricting it to a fixed polynomial structure, the model becomes theoretically broader and structurally more flexible. From a mathematical point of view, determining the Lie symmetry structure of the generalized equation is essential for understanding its reduction mechanisms and invariant properties. Accordingly, we establish a Lie symmetry classification with respect to the admissible forms of $f(u)$.

The obtained infinitesimal generators determine the corresponding Lie symmetry algebra and allow reductions of the governing partial differential equation to lower-dimensional equations. Making use of the optimal system of two-dimensional subalgebras of the symmetry algebra, we obtain reductions of the equation to ODEs for each of the canonical forms of $f(u)$. In particular, one of these reductions leads to a

traveling-wave form, through which the equation is reduced to an ordinary differential equation. This provides a link between the symmetry structure of the equation and the traveling-wave analysis.

For the exact solution analysis, the nonlinear form $f(u) = \alpha u^2 + \beta u^3$ is considered. Under the associated traveling-wave reduction, exact solutions are obtained in terms of Jacobi elliptic functions. In certain limiting cases, these solutions reduce to hyperbolic and trigonometric wave forms, indicating the existence of localized, singular, and periodic traveling-wave structures.

Finally, the reduced ordinary differential equation is rewritten as a planar dynamical system in order to investigate the stability and qualitative behavior of the traveling-wave solutions. The corresponding phase portraits are compared with the analytical wave profiles and are shown to clarify the occurrence of different solution types under various parameter regimes.

4. **Classification of KdV-type Bi-Hamiltonian systems**

by

Xiaoman Luo: Shandong University of Science and Technology, China & INFN, Italy

Abstract: The KdV-type Bi-Hamiltonian system has the following form: $P = P_1$ and $Q = Q_1 + R_k$, where P_1, Q_1 are homogeneous first-order Hamiltonian operators, R_k is a homogeneous k-st order Hamiltonian operator, and all operators are mutually compatible. Equations of this type include the KdV, AKNS, Kaup-Broer, Dispersive Water Waves and Dym equations. Conditions under which homogeneous operators of the first, second and third orders are Hamiltonian are well known. On the other hand, the compatibility of two operators is a way more complicated problem. In this talk, we illustrate the compatibility conditions and extensions between P_i and R_3. Then, we will use the classification of operators R_3 when the number of dependent variables is 3 and 4 to the aim of classifying trios in these dimensions.